MATH 312 LECTURE 1 : Gaussian Elimination.

→ Introduce myself.

→ Go over the syllabus:

1) Email / office hours.

2) Canvas

3) Background

- → www.pollev.com/math312
- → Previous lin. alg. (M240?)
- → Interests (major)

4) Evaluation: • Homework, weekly, starting this one! 25%!

↳ No late homework (two worst scores dropped)

• 1st Midterm exam → Before drop deadline!

5) Course overview: Book "Introduction to linear algebra", Gilbert Strang, 5th Ed.

Linear equations → Method to solve "Gaussian Elimination" (Chap. 2)

↓

Abstraction: deep understanding (vector spaces, linear combinations) (Chap. 3, (4))

↓

[Transition] (Determinants) (Chap. 5)

↓

The heart of real-life lin. algebra: Eigenvalues, eigenvectors and linear transf. (Chapt 6, 8)

↓

Applications (Chapt 7, 10)
- SVD → digital imaging
- PCA → machine learning
- Markov chains → Google PageRank
- ⋮

Chapter 2: Solving Linear Equations.

I. Algebra and Geometry in Equations.

(Sections 2.1, 2.2)

Example 1:

Let's start with a simple example: Solve for x and y the following systems of equations,

a) $\left.\begin{aligned} 4x + 7y &= 1 \\ x - 2y &= 4 \end{aligned}\right\}$ b) $\left.\begin{aligned} 4x + 7y &= 1 \\ y &= -1 \end{aligned}\right\}$ c) $\left.\begin{aligned} x &= 2 \\ y &= -1 \end{aligned}\right\}$

Solution:

Problem c) ✓, b) is also easy: $y = -1 \Rightarrow$

$\Rightarrow 4x - 7 = 1 \Rightarrow 4x = 8 \Rightarrow x = 2$.

But a)?

Several ways. We can try solving for x in the second equation first

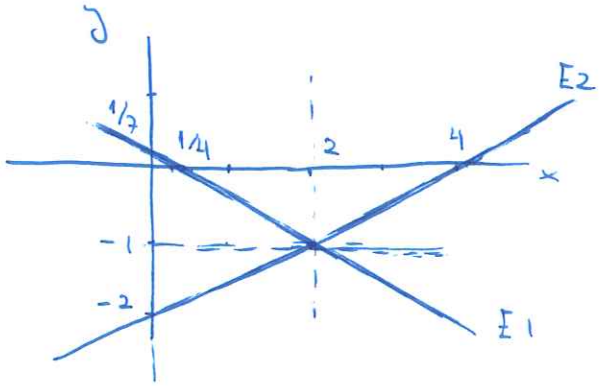
$x = 4 + 2y$, and substituting that into the first:

$4(4 + 2y) + 7y = 1 \Rightarrow 16 + 8y + 7y = 1 \Rightarrow 15y = -15 \Rightarrow y = -1$.

Or we can solve it geometrically:

Denote | E1 to $4x+7y=1$ (a line)

| E2 to $x-2y=4$ (another line)



E1: $x=0 \rightarrow y=1/7$

$y=0 \rightarrow x=1/4$

E2: $x=0 \Rightarrow y=-2$

$y=0 \rightarrow x=4$

Problem: it can be hard to draw and inaccurate.

II.| Gaussian Elimination.

II.1) The Algorithm

We look for a systematic approach.

Consider again

$$\begin{aligned} 4x+7y&=1 \quad E1 \\ x-2y&=4 \quad E2 \end{aligned}$$

We will try to go from this to c), and we will do it passing through b).

- First step: We try to eliminate the dependence on x in the second equation (E2).
  We do that using the "pivot" as ~~a~~ a division

pivot ↗

$$\left.\begin{array}{ll} \boxed{4}x + 7y = 1 & E1 \\ x - 2y = 4 & E2 \end{array}\right\} \xrightarrow{E2' = E2 - \frac{1}{4}E1} \left.\begin{array}{l} 4x + 7y = 1 \\ -\frac{15}{4}y = \frac{15}{4} \end{array}\right\} \rightarrow$$

$$-\frac{1}{4}\cdot E1 \rightsquigarrow -\frac{1}{4}4x - \frac{7}{4}y = \frac{-1}{4} \qquad -2-\frac{7}{4} = \frac{-15}{4}$$

$$E2 - \frac{1}{4}E1 \rightsquigarrow \cancel{x} - 2y - \cancel{x} - \frac{7}{4}y = 4 - \frac{1}{4} \qquad 4 - \frac{1}{4} = \frac{15}{4}$$

→ New system $\begin{array}{l} 4x + 7y = 1 \\ \quad\;\; y = -1 \end{array}$ (Triangular)

- Second step: Back substitution $y = -1$ into E1,

$$4x - 7 = 1 \Rightarrow x = 2 .$$

Method: Algorith for Gaussian Elimination

$(n = 1)$

→ 1) Rearrange your equations so that the n-th variable in the n-th equation has a non-zero coefficient (called PIVOT).

$n \leftarrow n+1$

2) Use the PIVOT to clear out ~~all~~ the n-th variable from all equations below the n-th one.

3) Is the system triangular? $\xrightarrow{Yes}$ Back substitution.

NO

Example: Solve for x, y, z in the following system

$$\left.\begin{array}{ll} E1 & x+y+z=4 \\ E2 & x-2y-2z=7 \\ E3 & 2x-3y+3z=-5 \end{array}\right\}$$

Q: Geometric interpretation of E1?

Solution:

Try it geometrically → intersection of planes → hard!

Try it ~~substitution~~ solving for z first and substituting ...
doable but kind of messy

Let's use "our" algorithm (gaussian elimination):

n = 1

1) Our 1-st variable (x) in the 1-st equation has coefficient 1 (NON-ZERO), so the pivot is 1.

2) Rewrite E2 and E3 without x:

$$E2' = E2 - E1 \longrightarrow 0 - 3y - 3z = 3$$

$$E3' = E3 - 2\cdot E1 \longrightarrow 0 - 5y + z = -13$$

New system:
$$\left.\begin{array}{rl} x + y + z &= 4 \\ ▨ \; -3y - 3z &= 3 \\ ▨ \; -5y + z &= -13 \end{array}\right\}$$
Not triangular yet.

(No x) ↗

Q: ask for pivot

Multiplier: $\dfrac{\text{entry to eliminate}}{\text{pivot}}$

n=2: The pivot is now -3 ($E2' \equiv -3y-3z=3$)

So $E3'' = E3' - \boxed{\dfrac{-5}{-3}} E2$, i.e.,

$\dfrac{-5}{-3} E2 = \dfrac{5}{3} E2 \equiv -5y-5z=5$ $\Big\}\rightarrow E3'' \equiv 0 + 6z = -18$

New system:

$$\left.\begin{array}{rl} \textcircled{1}x + y + z = 4 & E1 \\ \textcircled{-3}y - 3z = 3 & E2' \\ \textcircled{6}z = -18 & E3'' \end{array}\right\} \text{Triangular} \rightarrow$$

pivots

→ Easy to solve backwards:

First, use $E3''$ to get $z=-3$.

Second, substitute this in $E2'$ to get $-3y+9=3 \Rightarrow y=2$.

Third, substitute both z and y in $E1$ to get $x=5$.

So, the solution is $x=5, y=2, z=-3$.

(only in one point the three plane intersect!).

• Question: Can this algorithm go wrong?

↳ What happens if we find a ZERO instead of a pivot?

II.2) Breakdown of Elimination: Singular cases.

Let's see what happens in the following 3 systems of linear equations:

Case 1

$$\left.\begin{array}{l} x+y+z=4 \\ 2x+2y-2z=7 \\ 2x-y-z=11 \end{array}\right\}$$

Case 2

$$\left.\begin{array}{l} x+y+z=4 \\ x-2y-2z=7 \\ 2x-y-z=10 \end{array}\right\}$$

Case 3

$$\left.\begin{array}{l} x+y+z=4 \\ x-2y-2z=7 \\ 2x-y-z=11 \end{array}\right\}$$

- <u>Case 1</u>

Pivot → 1, so

$$\begin{array}{rl} x+y+z=4 & E1 \\ -4z=-1 & E2' \\ -3y-3z=3 & E3' \end{array}$$

Problem! Zero y-coefficient → no pivot

But, our algorithm solved this: step 1) Rearrange!

$$\left.\begin{array}{l} x+y+z=4 \\ -3y-3z=3 \\ -4z=-1 \end{array}\right\}$$ (Backwards...)

Remark: This is NOT a singular case. The algorithm works and we have one and only one solution.

- Case 2

After 1st iteration we obtain
$$\begin{aligned} x + y + z &= 4 \quad E1 \\ -3y - 3z &= 3 \quad E2' \\ -3y - 3z &= 2 \quad E3' \end{aligned}$$

Second iteration (!):
$$\begin{aligned} x + y + z &= 4 \quad E1 \\ -3y - 3z &= 3 \quad E2' \\ 0 &= -1 \quad E3'' \end{aligned}$$

Not possible! ⟶ $0 = -1$

↳ Our system has **no** solution : Singular case with zero solutions.

- Case 3

We get
$$\begin{aligned} x + y + z &= 4 \quad E1 \\ -3y - 3z &= 3 \quad E2' \\ -3y - 3z &= 3 \quad E3' \end{aligned} \quad \rightarrow \quad \begin{aligned} x + y + z &= 4 \quad E1 \\ -3y - 3z &= 3 \quad E2' \\ 0 &= 0 \quad E3'' \end{aligned}$$

Always true!

So our system is:

$$\left.\begin{aligned} x + y + z &= 4 \\ -3y - 3z &= 3 \end{aligned}\right\}$$ Less equations than ~~variables~~ unknowns ⇒

⇒ Infinite solutions (singular case).

[redacted]

<u>Remark</u>: In the singular case with infinite solutions, we might want to find one or two particular solutions, or better, the whole set of solutions.

How?

Since we have more variables than pivots, these extra variables act as parameters.

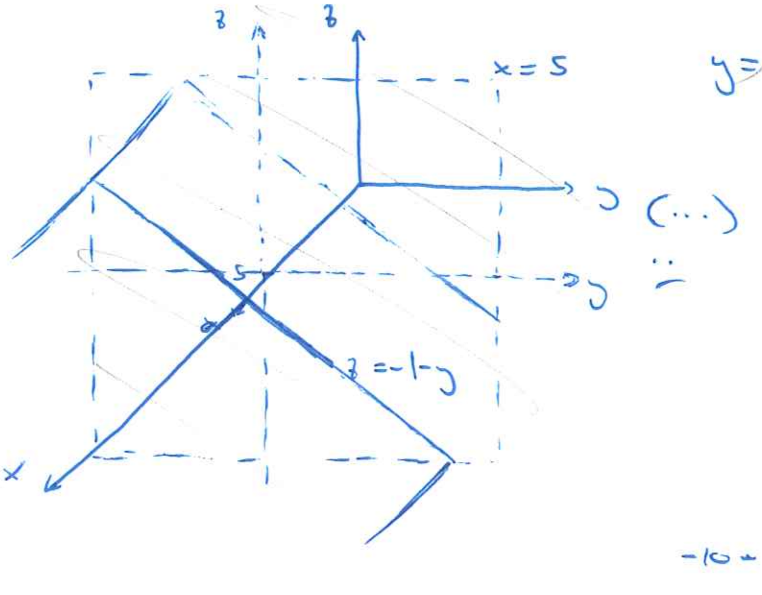
In our Case 3: Let $z \in \mathbb{R}$ be a parameter (or "free variable").

Then, $-3y = 3 + 3z \Rightarrow y = -1 - z$,

and $x + (-1 - z) + z = 4 \Rightarrow x = 5$.

So our set of solutions are: $\begin{array}{l} x = 5 \\ y = -1 - z \end{array}$ with $z \in \mathbb{R}$.

Geometrically, this is a line in the plane $x = 5$.

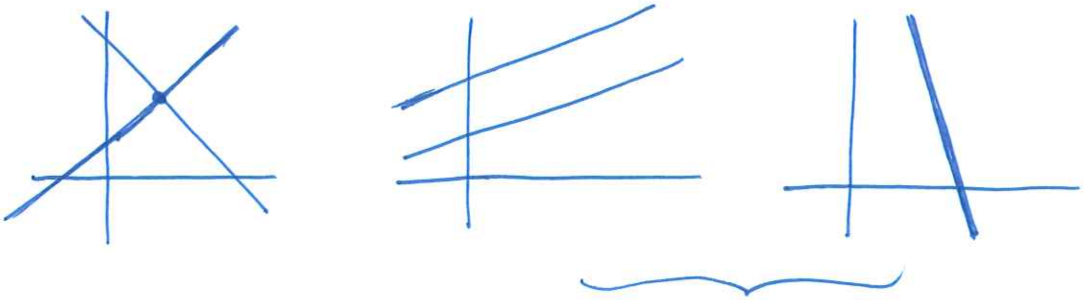


$y = -1 - z \Leftrightarrow z = -1 - y$

Important! In linear systems of equations, only those three possibilities: $\{0, 1, \infty\}$ solutions

Why?

We can see it in the 2 by 2 case thinking geometrically



Singular cases.

Can you think why in the n-dimensional case?

It will be easier after next lecture (algebra viewpoint).

MATH 312
LECTURE 2 : Matrices and inverses.

Last time: Elimination.

<u>Exercise:</u> (Elim. with parameters)

Let $a, b \in \mathbb{R}$. Solve for x and y:

$$\begin{aligned} ax + 2y &= 4, \\ x - y &= b. \end{aligned}$$

<u>Solution:</u> Q

1) <u>$a = 0$</u>

$$\left.\begin{aligned} 2y &= 4 \\ x - y &= b \end{aligned}\right\} \Rightarrow \begin{aligned} y &= 2 \\ x &= b + 2. \end{aligned}$$

(Only one solution; non-singular case)

(pivots? $p_1 = 1, p_2 = 2$)

2) <u>$a \neq 0$</u>

$$\begin{aligned} ax + 2y &= 4 \quad E1 \\ x - y &= b \quad E2 \end{aligned} \xrightarrow[E2' = E2 - \frac{1}{a}E1]{} \begin{aligned} ax + 2y &= 4 \quad E1 \\ \left(-1 - \frac{2}{a}\right)y &= b - \frac{4}{a} \quad E2' \end{aligned}$$

2.1) <u>$-1 - \frac{2}{a} = 0$</u> ($\equiv a = -2$)

$$\begin{aligned} -2x + 2y &= 4 \quad E1 \\ 0 &= b + 2 \quad E2' \end{aligned}$$

2.1.1) $b+2=0 \Rightarrow \left.\begin{array}{r} -2x+4y=4 \\ 0=0 \end{array}\right|$ Infinite Solutions.

$\downarrow$

2.1.2) $b+2 \neq 0 \rightarrow$ No solutions. $\rightarrow$ Singular cases

2.2) $\underline{a \neq -2}$

Then we have two pivots, $p_1 = a$, $p_2 = -1 - \frac{2}{a}$, and we can solve backwards,

$$y = \left(b - \frac{4}{a}\right) / \left(-1 - \frac{2}{a}\right) = \frac{\frac{ba-4}{\cancel{a}}}{-\frac{a+2}{\cancel{a}}} = \frac{4-ba}{a+2} \Rightarrow$$

$$\Rightarrow x = \frac{1}{a}\left(4 - 2\,\frac{4-ba}{a+2}\right) = \frac{1}{a}\,\frac{4a+\cancel{8}-\cancel{8}+2ba}{a+2} = \frac{4+2b}{a+2} \checkmark .$$

(only one solution; non-singular case).

Exercise: For which three values of a the following system is singular? (No se ha hecho en clase)

$ax + 2y + 3z = 1$ E1

$ax + ay + 4z = 2$ E2

$ax + ay + az = 3$ E3

Solution:

- Case $a = 0$: $2y + 3z = 1$

  $4z = 2$

  $0 = 3 \longrightarrow$ No solution (Singular case).

- Case $a \neq 0$:

After iteration 1

$ax + 2y + 3z = 1$ E1

$(a-2)y + z = 1$ E2'

$(a-2)y + (a-3)z = 2$ E3'

- Subcase $a = 2$:

$$\left.\begin{aligned} 2x + 2y + 3z &= 1 \quad E1 \\ z &= 1 \quad E2' \\ -z &= 2 \quad E3' \end{aligned}\right\} \xrightarrow{\text{Iteration 2}}$$

$2x + 2y + 3z = 1$ E1

$z = 1$ E2'

$0 = 3$ E3' $\rightarrow$ No solution (Singular case).

- Subcase $a \neq 2$: $ax + 2y + 3z = 1$ E1'

  $(a-2)y + z = 1$ E2' $\rightarrow$ 2nd iteration

  $(a-2)y + (a-3)z = 2$ E3'

$ax + 2y + 3z = 1$ E1'

$(a-2)y + z = 1$ E2'

$(a-4)z = 1$ E3'' $\Rightarrow$ If $a=4$, no solution (sing. case).

In summary,

$$\left.\begin{array}{l} a=0 \\ a=2 \\ a=4 \end{array}\right\} \Rightarrow \text{No solution (singular cases)}$$

other a $\Rightarrow$ Solution is $z = \frac{1}{a-4}$,

$$y = \left(1 - \frac{1}{a-4}\right)\frac{1}{a-2}$$

$$x = \left(1 - \frac{3}{a-4} - \frac{2}{a-2}\left(1 - \frac{1}{a-4}\right)\right)\frac{1}{a}$$

III.1 Matrices (~ sec. 2.4)

A system like

$$\begin{aligned} 3x - 4y + 5z &= 10 \quad E1 \\ 4x + y - z &= 12 \quad E2 \\ 2x + 2y + 3z &= 1 \quad E3 \end{aligned}$$

can be writen as a matrix:

$$\begin{array}{c} \\ eq1 \\ eq2 \\ eq3 \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc:c} 3 & -4 & 5 & 10 \\ 4 & 1 & -1 & 12 \\ 2 & 2 & 3 & 1 \end{array}\right] \end{array}$$

(The so-called Augmented Matrix).

So now Gaussian Elim. is done through row operations; we would start as follows

$$\begin{aligned} R2' &= R2 - \frac{4}{3}R1 \\ R3' &= R3 - \frac{2}{3}R1 \end{aligned} \rightarrow \begin{bmatrix} \enclose{circle}{3} & -4 & 5 & 10 \\ 0 & \enclose{circle}{19/3} & \cdots & \\ 0 & 14/3 & \cdots & \end{bmatrix} \cdots$$

Just a matter of notation!

III.1) Matrix multiplication

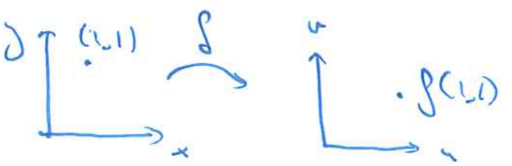
We all know for $A = \begin{bmatrix} 3 & 2 \\ 6 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ that

$AB = \begin{bmatrix} 8 & -1 \\ 7 & 16 \end{bmatrix}$, $BA = \begin{bmatrix} 12 & -1 \\ -9 & 12 \end{bmatrix}$ (<u>Note</u>: in general, $AB \neq BA$!!) (*)

But why do matrix multiply like this?

Let's think of the system $\left.\begin{matrix} 3x + 2y = u \\ 6x - 5y = v \end{matrix}\right]$

Here, given (x, y) we obtain (u, v). That is, we have a "formula", or a function "f":



$f(1,1) = (3+2, 6-5) = (5,1) \underset{\text{general}}{\rightsquigarrow} (u,v) = f(x,y) = (3x+2y, 6x-5y)$.

- This is of course linear:

<u>**LINEAR**</u>: 1) $f(ax, ay) = (3ax + 2ay, 6ax - 5ay) = a f(x,y)$.

2) $f(x+\tilde{x}, y+\tilde{y}) = f(x,y) + f(\tilde{x}, \tilde{y})$.

(*) From a question in class: There are indeed some matrices which do commute (of course!). For example:

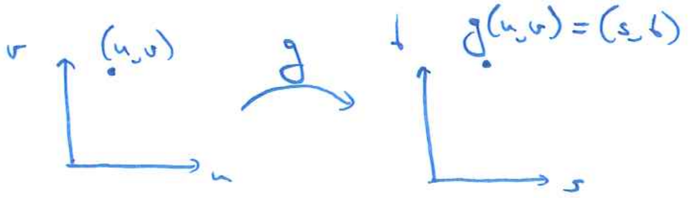
1) The identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and any matrix A: $AI = IA$

2) Two diagonal matrices $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$

But there is not a simple criteria to define all matrices that commute. So, in principle, given two matrices, we cannot commute them unless we explicitly check it ✓.

Imagine that we have another function that takes this (u,v) to a new point (s,t):

$$\left.\begin{aligned} s &= 2u + v \\ t &= u - 2v \end{aligned}\right\}$$



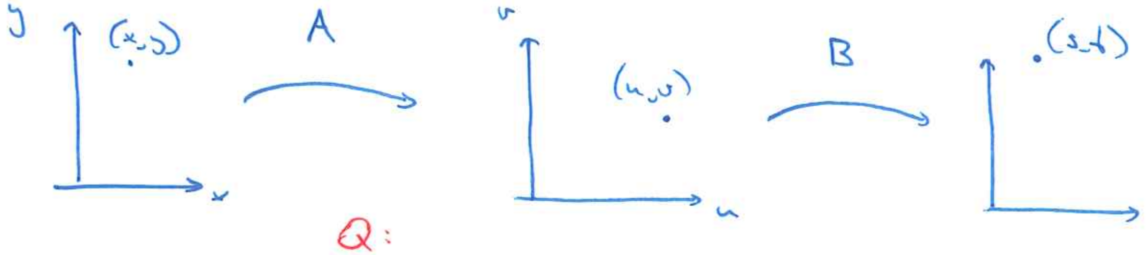
- **Question**: Given (x,y), what is (s,t)?

$s = \ldots\, x + \ldots\, y$
$t = {}^{M}\ldots\, x + \ldots\, y$

Well,

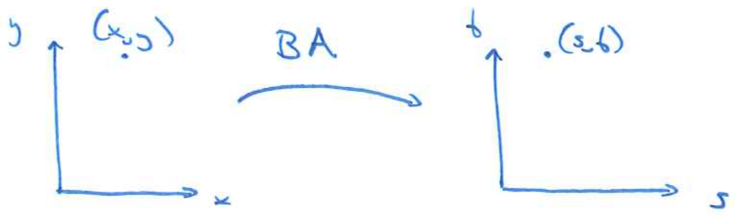
$$s = 2u+v \xrightarrow[v=6x-5y]{u=3x+2y} s = 2(3x+2y)+(6x-5y) \rightsquigarrow$$
$$t = u-2v \xrightarrow{\qquad} t = (3x+2y)-2(6x-5y)$$

$$\rightsquigarrow \begin{aligned} s &= (2\cdot 3 + 1\cdot 6)x + (2\cdot 2 - 1\cdot 5)y = 12x - y \\ t &= (3\cdot 1 - 2\cdot 6)x + (1\cdot 2 - 2\cdot(-5))y = -9x + 12y \end{aligned} \quad !!$$



Q:

so we want that BA acts to give



(Again, clearly the order is important, $AB \neq BA$!!)

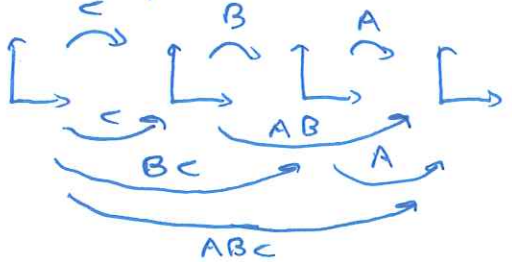
- Matrix multiplication is composition of linear functions!

$$\begin{bmatrix}u\\v\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix},\quad \begin{bmatrix}s\\t\end{bmatrix}=B\begin{bmatrix}u\\v\end{bmatrix}\longrightarrow\begin{bmatrix}s\\t\end{bmatrix}=BA\begin{bmatrix}x\\y\end{bmatrix}$$

$$(u,v)=f(x,y)\quad,\quad(s,t)=g(u,v)\longrightarrow(s,t)=(g\circ f)(x,y)$$

From here we see that matrix multiplication is associative:

$$ABC=(AB)C=A(BC).$$



- Important facts:

1) Identity matrix $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}\rightarrow AI=IA=A\quad\forall A.$

2) Ways of multiplying matrices:

$$\begin{array}{cc} & n \\ m & \begin{bmatrix}a_{11}&a_{12}\cdots&a_{1n}\\a_{21}&a_{22}\cdots&a_{2n}\\\vdots&\vdots\quad\ddots&\vdots\\a_{m1}&a_{m2}\cdots&a_{mn}\end{bmatrix} \\ & A\end{array}\begin{array}{cc} p & \\ \begin{bmatrix}b_{11}&\cdots&b_{1p}\\\vdots&&\\b_{n1}&\cdots&b_{np}\end{bmatrix} & n \\ B & \end{array}=\begin{array}{cc} & p \\ m & \begin{bmatrix}c_{11}&\cdots&c_{1p}\\\vdots&\ddots&\vdots\\c_{m1}&\cdots&c_{mp}\end{bmatrix}\end{array}$$

→ General formula: $\boxed{c_{ij}=\sum_{k=1}^{n}a_{ik}\cdot b_{kj}}\equiv$ dot product of row i with column b

↳ nº of columns of A has to be equal to nº of rows of B!

→ By columns: $A\left[\begin{array}{c|c|c|c} b_1 & b_2 & \cdots & b_p \end{array}\right] = \left[\begin{array}{c|c|c|c} Ab_1 & Ab_2 & \cdots & Ab_p \end{array}\right]$.

→ By rows: $\left[\begin{array}{c} a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_m \end{array}\right] B = \left[\begin{array}{c} a_1 B \\ \hline a_2 B \\ \hline \vdots \\ \hline a_m B \end{array}\right]$.

Example:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ (here $m=4, n=3, p=2$) ($C = AB$ has 4 rows, 2 columns)

- Classic way: $AB = \begin{bmatrix} a_1 \cdot b_1 & a_1 \cdot b_2 \\ a_2 \cdot b_1 & a_2 \cdot b_2 \\ a_3 \cdot b_1 & a_3 \cdot b_2 \\ a_4 \cdot b_1 & a_4 \cdot b_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 7 \\ 1 & 1 \end{bmatrix}$

- By columns:

$$Ab_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad Ab_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 1 \end{bmatrix}$$

- By rows:

$$a_1 B = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \ldots$$

→ Block Multiplication: (do example first)

Now we split matrices into "blocks"

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

where A_{ij} is an $m \times n$ matrix and B_{jk} an $n \times p$ matrix.
We can multiply as if A_{ij} and B_{jk} were numbers:

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} & A_{11}B_{13} + \dots \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

→ Ex:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \hline 0 & 1 \\ 3 & 4 \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \\ 6 & 8 \\ \hline 1 & 3 \\ 6 & 8 \end{array}\right]$$

→ Important particular case: Columns × Rows.

Consider

$$A = \left[\begin{array}{c|c|c} a_1 & \dots & a_n \end{array}\right], \quad B = \begin{bmatrix} b_1 \\ \hline \vdots \\ \hline b_n \end{bmatrix}$$

and look at a_i, b_j as blocks
(i.e., they can be vectors, matrices or just numbers!).

The result is: Q.

$$AB = \left[\begin{array}{c|c|c} a_1 & \cdots & a_n \end{array}\right] \left[\begin{array}{c} b_1 \\ \hline \vdots \\ \hline b_n \end{array}\right] = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Let's see it with two examples:

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$$AB = \left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 1 \\ \hline 0 & 2 & 1 \end{array}\right] = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 2 \\ 0 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 3 & 8 & 7 \end{bmatrix}$$

- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$AB = \left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array}\right] \left[\begin{array}{c} x_1 \\ \hline x_2 \end{array}\right] = \begin{bmatrix} 1 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 4 \end{bmatrix} x_2$$

Linear combination of the columns of A !!

III.2) The column picture of linear systems.

Using matrix algebra, linear systems can now be written as

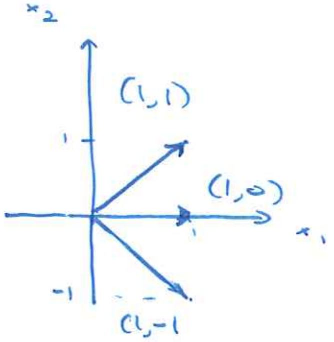
$Ax = b$, for example,

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponds to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 0 \end{cases}$$

And we can see it as linear comb.:

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



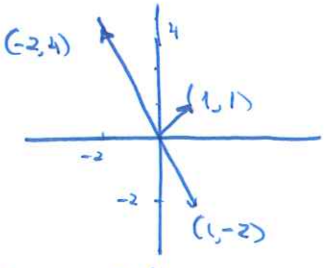
x_1, x_2 are the coefficients of the linear combination between $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to give $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$!

- Q: Can you imagine what does breakdown of elimination looks like? ☺

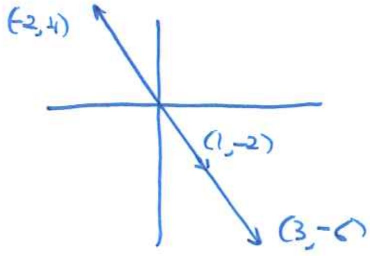
Example:

$$\left.\begin{aligned} x - 2y &= 1 \\ -2x + 4y &= 1 \end{aligned}\right\} \qquad \left.\begin{aligned} x - 2y &= 3 \\ -2x + 4y &= -6 \end{aligned}\right\}$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} x + \begin{bmatrix} -2 \\ 4 \end{bmatrix} y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -2 \end{bmatrix} x + \begin{bmatrix} -2 \\ 4 \end{bmatrix} y = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$



No solution



Infinite solutions.

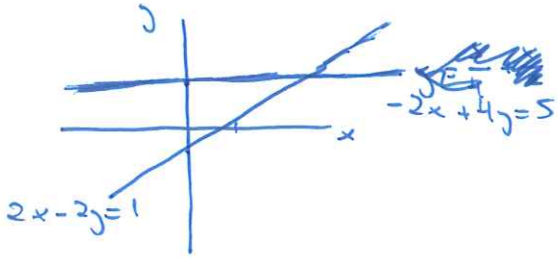
Summary of linear systems:

$$\boxed{\begin{aligned} 2x - 2y &= 1 \\ -2x + 4y &= 5 \end{aligned}} \quad = \quad \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Linear system

Matrix multiplication

$$\Vert \qquad\qquad\qquad \Vert$$



$$=$$

$$x \begin{bmatrix} 2 \\ -2 \end{bmatrix} + y \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Intersections of lines
(hyperplanes)

Linear combinations.

IV. Inverse Matrices (~2.5) Q: Break? (→ No)

Def.: The matrix A has an inverse, denoted A^{-1}, if

$$AA^{-1} = A^{-1}A = I$$

(Note: A has to be square)

IV.1) Gauss-Jordan

Question: Which matrices have an inverse? (NOT All!)
How to compute A^{-1}?

Let's try an example:

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \to A^{-1}? \quad \text{Denote } A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \; x_{11}, x_{12}, x_{21}, x_{22}?$$

$$AA^{-1} = I \Rightarrow \underbrace{\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}}_{A} \left[\begin{array}{c:c} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow$$

$$\Rightarrow \left[\begin{array}{c:c} \underbrace{\begin{matrix} 4x_{11} + 2x_{21} \\ x_{11} + x_{21} \end{matrix}}_{A\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}} & \underbrace{\begin{matrix} 4x_{12} + 2x_{22} \\ x_{12} + x_{22} \end{matrix}}_{A\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}} \end{array}\right] = \left[\begin{array}{c:c} 1 & 0 \\ 0 & 1 \end{array}\right] \Rightarrow \text{Two linear systems} \to$$

$$1)\ \underbrace{\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , 2)\ \underbrace{\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(← We stopped here approx.)

We can solve the first, for example:

$$\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sim \left[\begin{array}{cc:c} 4 & 2 & 1 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{E2' = E2 - \frac{1}{4}E1}$$

$$\left[\begin{array}{cc:c} 4 & 2 & 1 \\ 0 & 1/2 & -1/4 \end{array}\right] \underset{E2'' = 2E2}{\sim} \left[\begin{array}{cc:c} 4 & 2 & 1 \\ 0 & 1 & -1/2 \end{array}\right] \underset{E1' = E1 - 2E2}{\sim} \left[\begin{array}{cc:c} 4 & 0 & 2 \\ 0 & 1 & -1/2 \end{array}\right] \underset{E1'' = \frac{E1'}{4}}{\sim}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{array}\right] \rightarrow \begin{array}{l} x_{11} = 1/2 \\ x_{21} = -1/2 \end{array} .$$

We can do the same for system 2); indeed, at the same time

$$\cancel{\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}} \quad \left[\begin{array}{cc:cc} \underbrace{\begin{matrix} 4 & 2 \\ 1 & 1 \end{matrix}}_{A} & \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \end{array}\right] \underset{\substack{\uparrow \\ \text{Row} \\ \text{operations}}}{\sim} \left[\begin{array}{cc:cc} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} 1/2 & \cdots \\ -1/2 & \cdots \end{matrix}}_{A^{-1}} \end{array}\right]$$

(first: Gaussian Elimination
Then: "Gaussian" elim. upwards, and dividing by the pivots to get 1s).

- Remark: The matrix A has an inverse if and only if A has n pivots ($A \equiv n\times n$ matrix).

- Remark: Notice that by solving 2 linear systems

$Ax_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $Ax_2 = \begin{bmatrix}0\\1\end{bmatrix}$, we have solved

$Ax_3 = b$ for no matter which b !! ($b = \begin{bmatrix}b_1\\b_2\end{bmatrix}$)

This is the core of linearity:

$$b = b_1\begin{bmatrix}1\\0\end{bmatrix} + b_2\begin{bmatrix}0\\1\end{bmatrix} \to \left\| x_3 = b_1x_1 + b_2x_2 \right\|$$

$$\boxed{\begin{matrix}\text{Solution of}\\ Ax_3 = b\end{matrix}} = b_1 \boxed{\begin{matrix}\text{Solution of}\\ Ax_1 = \begin{bmatrix}1\\0\end{bmatrix}\end{matrix}} + b_2 \boxed{\begin{matrix}\text{Solution of}\\ Ax_2 = \begin{bmatrix}0\\1\end{bmatrix}\end{matrix}}$$

$$x_3 = A^{-1}b \qquad x_1 = \underbrace{A^{-1}\begin{bmatrix}1\\0\end{bmatrix}}_{\text{First column of } A^{-1}} \qquad x_2 = \underbrace{A^{-1}\begin{bmatrix}0\\1\end{bmatrix}}_{\text{Second column of } A^{-1}}$$

- Question (home):

Assume x_1 is the solution of $Ax_1 = e_1$, and x_2 that of $Ax_2 = e_2$. What is the solution of $Ax = b$ as a linear combination of x_1 and x_2?

Sol: $b = b_1e_1 + b_2e_2$ and we know that $e_1 = Ax_1$, $e_2 = Ax_2 \to$

$\to b = b_1Ax_1 + b_2Ax_2 = A(\underbrace{b_1x_1 + b_2x_2}_{= x})$ //.

IV. 2) Some important properties.

1) $(A^{-1})^{-1} = A$

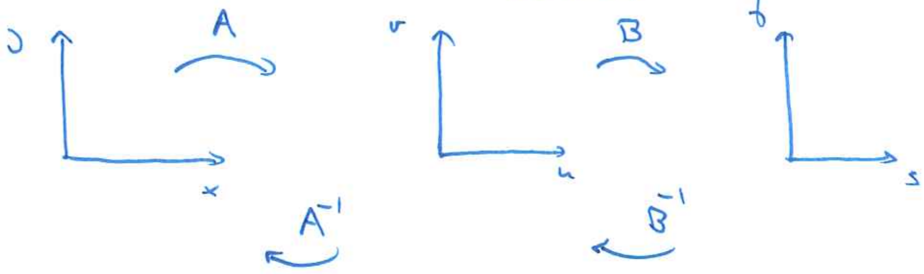
Q.: 2) $(AB)^{-1} = B^{-1}A^{-1}$ 3) $(A+B)^{-1} \neq A^{-1}+B^{-1}$ (Ex: $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$)

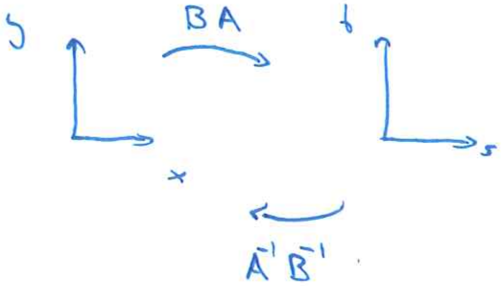
↳ $\nexists (A+B)^{-1}$!!

Why? Let's call $C=(AB)^{-1}$. We need that

$$\begin{matrix} C(AB)=I \\ (AB)C=I \end{matrix} \xrightarrow{\text{Then}} \begin{matrix} B^{-1}A^{-1}(AB)=B^{-1}(A^{-1}A)B=B^{-1}B=I, \\ (AB)B^{-1}A^{-1}=A(BB^{-1})A^{-1}=AA^{-1}=I \,// \end{matrix}$$

• **Remark**: Remember matrices as "functions" (linear)





3) The inverse (when exists) is unique.

(Home) • Why? Assume A has two different inverses, B and C ($B \neq C$).

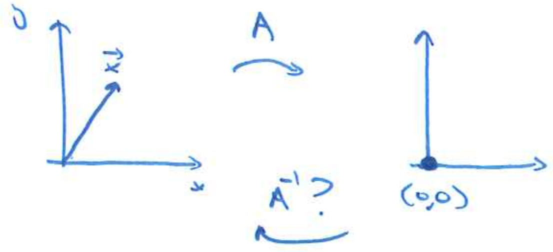
Then:

$$\left.\begin{matrix} B(AC)=BI=B \\ (BA)C=IC=C \end{matrix}\right\} \xrightarrow{B(AC)=(BA)C} B=C \;\lightning.$$

4B) (First "taste" of Nullspace)

- Q.: If $Ax = 0$ for some $x \neq 0$, does A^{-1} exist?

Clearly not: $A^{-1}Ax = A^{-1}0 \Rightarrow x = 0 \nleftarrow$.



That is:

- If A^{-1} exists, then $Ax = 0$ has only the solution $x = 0$.

|||

- If $\exists x \neq 0 \,/\, Ax = 0$, then A^{-1} does not exist.

MATH 312 LECTURE 3 : LU decomposition (I)

(Summary week 1) (IV) Inverse Matrices (~Sec. 2.5.)).

IV.1) Gauss-Jordan

Method: Given A an n×n matrix,

1) Create the augmented matrix given by [A I].

Ex. $A=\begin{bmatrix}4&2\\1&1\end{bmatrix} \rightsquigarrow \begin{bmatrix}4&2&1&0\\1&1&0&1\end{bmatrix}$

2) Perform Gaussian elimination to get the upper triangular part.

Ex. $\begin{bmatrix}4&2&1&0\\1&1&0&1\end{bmatrix} \rightsquigarrow \begin{bmatrix}4&2&1&0\\0&1/2&-1/4&1\end{bmatrix}$

3) Are there n pivots? → Yes: Go to step 4)

→ No: The inverse does NOT exist.

4) Continue with "Gaussian" elimination upwards to get the so-called "~~reduced echelon~~" form of A:

Ex. $\begin{bmatrix}4&2&1&0\\0&1/2&-1/4&1\end{bmatrix} \rightsquigarrow \begin{bmatrix}4&0&2&-4\\0&1/2&-1/4&1\end{bmatrix}$

5) Divide each row by its pivots to get A^{-1} !! (then we get the "reduced row echelon form" of A)

Ex. $\begin{bmatrix}4&0&2&-4\\0&1/2&-1/4&1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1&0&1/2&-1\\0&1&\underbrace{-1/2}&\underbrace{2}\end{bmatrix}$

Note: If A^{-1} exists, the RREF ("reduced...") is the identity matrix.

$$A^{-1}=\begin{bmatrix}1/2&-1\\-1/2&2\end{bmatrix}$$

- What is the method really doing?

Given A, we want to find A^{-1}. Denote $x_1, \dots, x_n$ its columns:

$A^{-1} = \left[x_1 \,|\, x_2 \,|\, \cdots \,|\, x_n \right]$. Then, we want to find these $x_1, \dots, x_n$ /

$$AA^{-1} = I \longrightarrow A\left[x_1 \,|\, \cdots \,|\, x_n \right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \cdots & 0 \\ 0 & \vdots & & 1 \end{bmatrix} = \left[e_1 \,|\, \cdots \,|\, e_n \right]$$

So we can solve n systems of equations, one per column:

$$\left.\begin{array}{l} Ax_1 = e_1 \\ \quad \vdots \\ Ax_n = e_n \end{array}\right\} \begin{array}{l} \rightsquigarrow x_1 \text{ (first column of } A^{-1}) \\ \quad \vdots \\ \rightsquigarrow x_n \text{ (last column of } A^{-1}) \end{array}$$

→ Gauss-Jordan does all this at the same time using the augmented matrix.

→ If there are not n pivots, at least one of the n systems won't have a solution (i.e., the inverse won't exist) ✓.

• <u>Remark</u>:

1) If A^{-1} exists, the system $Ax=b$ has one and only one solution for any b:

$$Ax = b \rightarrow A^{-1}Ax = A^{-1}b \rightarrow \underline{\underline{x = A^{-1}b}}.$$

$$\left(\begin{array}{c} A \text{ singular} \equiv A^{-1} \text{ doesn't exist} \\ \Updownarrow \\ Ax=b \text{ have } 0 \text{ or } \infty \text{ solutions.} \end{array}\right)$$

2) However, in practice, the inverse is never computed to solve a linear system: we simply apply Gaussian elimination.

Reason: Finding the inverse required solving n systems! .

2nd reason: In real-life situations, large matrices (~1 million rows) are usually "sparse" (most entries are zeros), so Gaussian elimination is very fast.
However, the inverse of a sparse matrix is usually "dense" (most entries not zero).

(3) Linearity → page -27-) • Q.:

IV.2) Some important properties → pages -28-, -29-.

V| Elementary matrices and elimination. (~Sec. 2.3)

Gauss-Jordan: $[A \;\; I] \xrightarrow[\text{operations.}]{\text{row}} [I \;\; A^{-1}]$.

This is the same as : $[A \;\; I] \xrightarrow[A^{-1}]{} [I \;\; A^{-1}]$.

Conclusion: Row operations can be encoded as multiplications by certain matrices.

- Three types of row operations:

T1 : Subtract l_{ij} times equation j from equation i.

T2 : Interchange row i and row j.

T3 : Scale row i by some number (only in Gauss-Jordan).

Which matrices do these operations?

1) T1 operation

Consider for example $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Let's do the first step in gaussian elimination:

$$R2' = R2 - \overset{\text{multiplier}}{\left(\frac{4}{2}\right)} R1$$

$$\hookrightarrow \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

We can check that

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

> In general, the matrix for a TI operation is the identity matrix with an extra nonzero entry given by $-l_{ij}$ (minus the multiplier) in the (i,j) position.

__Example:__ Write all the steps in Gaussian elimination as elementary matrices.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 2 & -3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 2 & -3 & 3 \end{bmatrix} \xrightarrow{R2' = R2 - R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 2 & -3 & 3 \end{bmatrix} \xrightarrow{R3' = R3 - 2R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & -5 & 1 \end{bmatrix} \rightarrow$$

$$\xrightarrow{R3'' = R3' - \frac{-5}{-3} R2'} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 6 \end{bmatrix}$$; the elementary matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5/3 & 1 \end{bmatrix}.$$

Remark: We have gone from A to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 6 \end{bmatrix} = U$, i.e.,

$$A \xrightarrow{E_{21}} \xrightarrow{E_{31}} \xrightarrow{E_{32}} U,$$ or as matrix multiplications,

$$E_{32} E_{31} E_{21} A = U \quad (U \equiv \text{upper triangular}).$$

- **Question**: Can we always invert these E_{ij} matrices?

Of course! Remember:
$$\begin{tikzcd} \text{↳} \arrow[r, bend left, "A"] & \text{↳} \arrow[l, bend left, "A^{-1}"] \end{tikzcd} \qquad AA^{-1} = A^{-1}A = I.$$

E_{ij} does the following: subtracts ℓ_{ij} times eq. j from eq. i.

E_{ij}^{-1} does the opposite effect: adds " " " " " " .

So, in the examples:

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5/3 & 1 \end{bmatrix}.$$

(→ Good enough if we get here).

2) **T2 operation** : The matrix that interchanges rows i and j is the identity matrix with the rows i and j interchanged, P_{ij}.

Ex:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_{23}} \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

↖ ↗

eqns 2 and 3 interchanged.

- **Question:** What is the inverse of P_{ij}? Easy...! $P_{ij}^{-1} = P_{ij}$ ✓.

3) **T3 operation:** The matrix that multiplies row i by a number λ is the identity with its i row multiplied by λ.

Ex:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_2} \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 12 & 0 & 3 \\ 1 & 0 & 3 \end{bmatrix} \leftarrow 2^{nd} \text{ eqn. multiplied by } 3.$$

Its inverse? $M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ //.

MATH 312
LECTURE 4
: LU decomposition (II) // Vector Spaces.

→ Finish elementary matrices (page -36-) (VI Elementary matrices and elimination (~ Sec 2.3)).

VI | LU factorization (~ sec 2.6)

Consider the example from last lecture:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 2 & -3 & 3 \end{bmatrix}$$

We did

$$A \xrightarrow[E_{21}]{R2' = R2 - R1} \cdots \xrightarrow[E_{31}]{R3' = R3 - 2R1} \cdots \xrightarrow[E_{32}]{R3'' = R3' - \frac{5}{3} R2'} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 6 \end{bmatrix}}_{\substack{U \\ \text{(Upper triangular)}}}$$

That is,

$$E_{32} E_{31} E_{21} A = U.$$

And we know the inverses of the elementary matrices! So,

$$A = (E_{32} E_{31} E_{21})^{-1} U = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} U = \underbrace{L}_{\substack{\text{Lower} \\ \text{triangular}}} \underbrace{U}_{\substack{\text{upper} \\ \text{triangular.}}}$$

- Remark: E_{ij} is always lower triangular. The inverse of a lower triangular is also lower triangular. The product of lower triangular matrices is also a lower triangular matrix.

Let's check it:

$$E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5/3&1\end{bmatrix}=$$

$$=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&5/3&1\end{bmatrix}=\begin{bmatrix}1&0&0\\\textcircled{1}&1&0\\\textcircled{2}&\textcircled{5/3}&1\end{bmatrix}$$ lower triangular.

These are the multipliers!
And they are in their position!

Q.: When A=LU exists?

<u>Conclusion</u>: If no row exchanges are needed a matrix A can be factorized as

$$A = \underbrace{L}_{\text{lower triangular}}\ \underbrace{U}_{\text{upper triangular}}\ ,\ \text{where}$$

$U \equiv$ result of doing Gaussian Elimination on A.

$$L=\begin{bmatrix}1&0&\cdots&0\\l_{21}&1&\cdots&0\\l_{31}&l_{32}&\ddots&\vdots\\\vdots&\vdots&\cdots&1\end{bmatrix}$$

multipliers.

("subtract from row2 $\textcircled{l_{21}}$ × row1",
"subtract from row3 $\textcircled{l_{31}}$ × row1",
"subtract from row3 $\textcircled{l_{32}}$ × row2",
...
"subtract from row i $\textcircled{l_{ij}}$ × row j")

position(i,j)

Remark: Some people prefer the more "symmetric" factorization

$A = LDU$, where

No visto en clase

$L \equiv$ same as before.

$D \equiv$ diagonal matrix containing the pivots, $D = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}$ (in 3d case)

$U \equiv$ This is the previous U but each row is divided by its pivot (thus it only has 1 on the diagonal).

Ex: (Previous one).

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \\ 2 & -3 & 3 \end{bmatrix}$$

1) LU factorization: $A = LU$, with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5/3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 6 \end{bmatrix}$$

2) LDU factorization:

$$A = LDU, \text{ with } L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5/3 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- Important: How to solve $Ax=b$ for many different b?

- Easy answer: Compute A^{-1}, then $x = A^{-1}b$ (bad option in real-life).
- Good one: Use $A=LU$ as follows,

$A = LU \rightsquigarrow \underbrace{LUx}_{\text{vector}} = b$. Denote $y = Ux$.

Then, we just need to solve two linear systems in order:

1) $Ly = b \rightsquigarrow y$. (Easy! Since L is triangular, we solve by substituting "forwards")

2) Now that we know y, solve $Ux = y \rightsquigarrow x$ ✓.
(Easy too! Substitute "backwards".)

VII | Transposes and permutations.

Well-Known things:

1) A^T : If $A = \left[a_1 \,\middle|\, \cdots \,\middle|\, a_n \right] \rightsquigarrow A^T = \begin{bmatrix} - & a_1 & - \\ & \vdots & \\ - & a_n & - \end{bmatrix}$ In index notation: $A = [a_{ij}] \rightsquigarrow A^T = [a_{ji}]$

((Columns become rows, obviously, rows become columns).

2) $(A^T)^T = A$

3) $(A+B)^T = A^T + B^T$ $\left(\text{But } (A+B)^{-1} \neq A^{-1} + B^{-1} \right)$

4) $(A^T)^{-1} = (A^{-1})^T$

5) Inner product: Notice that if x and y are column vectors, the inner product $x \cdot y$ can be written using the transpose:

$$x \cdot y = x^T y = y^T x$$

$$\left(x \cdot y \text{ is a scalar, thus } (x \cdot y)^T = x \cdot y \Rightarrow \underbrace{(x^T y)^T}_{} = x^T y \right.$$
$$\left. = y^T (x^T)^T = y^T x \right).$$

6) Symmetric matrix : $S^T = S$.

7) Given a matrix A, $A^T A$ is always symmetric.

8) If A is symmetric, then $A = LDU$ becomes $A = LDL^T$.

$\left(A = A^T \Rightarrow LDU = U^T D^T L^T = U^T D L^T \Rightarrow U = L^T \right)$.

- Permutation matrix (PA = LU)

Def. A permutation matrix P has the rows of the identity in any order.

Basically, what they do is to change the position of rows when multiply:

Ex. $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to P_{12} \left[\begin{array}{c} a_1 \\ \hline a_2 \\ \hline a_3 \end{array}\right] = \left[\begin{array}{c} a_2 \\ \hline a_1 \\ \hline a_3 \end{array}\right]$.

Recall that $P_{12}^{-1} = P_{12}$.

Notice that

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ is a permutation matrix too, but requires two exchanges:}$$

$$\overset{\shortparallel}{P_{32}P_{21}} \to (P_{32}P_{21})^{-1} = P_{21}^{-1}P_{32}^{-1} = P_{21}P_{32} \neq P_{32}P_{21}$$

Application: If a matrix needs row exchanges when applying Gaussian elimination, we first reorder the rows:

A ⟶ PA; this new matrix doesn't need row exchanges, so we can find PA = LU.

[redacted]

[Remark: If it doesn't require row exchanges, P = I].