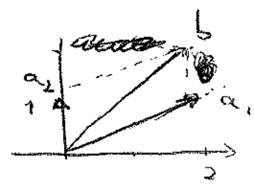
Chapter 3 : Vector Spaces and Subspaces.

I) Vector spaces and subspaces ($\sim$ Sec 3.1)

Recall the column picture:

$$Ax=b \sim \begin{bmatrix}2&0\\1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$$ is written as

$$\underbrace{\begin{bmatrix}2\\2\end{bmatrix}}_{b} = x_1\underbrace{\begin{bmatrix}2\\1\end{bmatrix}}_{a_1} + x_2\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{a_2}$$



We "see" that the system has a solution if b can be written as a "linear combination" of a_1 and a_2

This will remain true in "n" dimensions.

The "space" $\mathbb{R}^n$ will be $\mathbb{R}^n = \left\{ \vec{u} = \begin{bmatrix}u_1\\ \vdots \\ u_n\end{bmatrix} \text{ with } u_i \text{ in } \mathbb{R} \right\}$,

together with the usual operations:

1) Addition $\vec{u}+\vec{v} = \begin{bmatrix}u_1+v_1\\ \vdots \\ u_n+v_n\end{bmatrix}$ (the result is a vector).

2) Multiplication by scalar $\lambda\vec{u} = \begin{bmatrix}\lambda u_1\\ \vdots \\ \lambda u_n\end{bmatrix}$ (" " " " ").

Def: A linear combination of the vectors $\vec{u}_1,\dots,\vec{u}_n$ is

$\lambda_1\vec{u}_1 + \lambda_2\vec{u}_2 + \dots + \lambda_n\vec{u}_n$, with $\lambda_1,\dots,\lambda_n$ scalars.
(for us, $\mathbb{R}$)

• Def: Vector Space

A vector space, V, is a collection of objects (called "vectors") such that: any linear combination of objects in V also belong to V.

That is, if $u, v \in V$, then $\alpha u + \beta v \in V$ for any α, β scalars.

• Def: Subspace

A subspace V' of V is a subcollection of objects from V which itself forms a vector space.

→ Remark: One has to define what is the sum "+" and the scaling "$\lambda\cdot$". We will only consider the natural ones.

Examples:

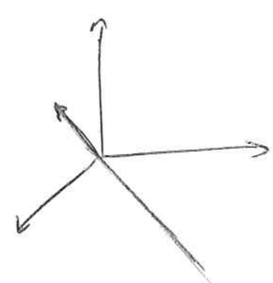
1) $\mathbb{R}^n$ is a vector space.

2) $\mathbb{R}^3$ in particular is a vector space.

→ The vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ itself is a subspace.

→ Any line through the origin.

→ Any plane through the origin.



3) Remark: All vector spaces must contain the zero vector.

↳ Since $\underbrace{0 \cdot \vec{u}}_{\text{lin. combination}} = \vec{0}$

(⇝ Paramos por aquí)

4) Polynomials form a vector space: The "vectors" are the polynomials.

$p(x), g(x) \rightsquigarrow c_1 p(x) + c_2 p(x)$ is still a polynomial.

numbers

Q. 5) Polynomials with degree 3 or less: It is a subspace of the polynomials.

$$\left.\begin{array}{l} p(x) \text{ degree } \leq 3 \\ g(x) \;\;\text{"}\;\; \leq 3 \end{array}\right\} \Rightarrow c_1 p(x) + c_2 g(x) \text{ is polynomial with degree at most } 3.$$

Q. c) But: polynomials with degree 3: <u>NOT</u> a subspace.

Why:

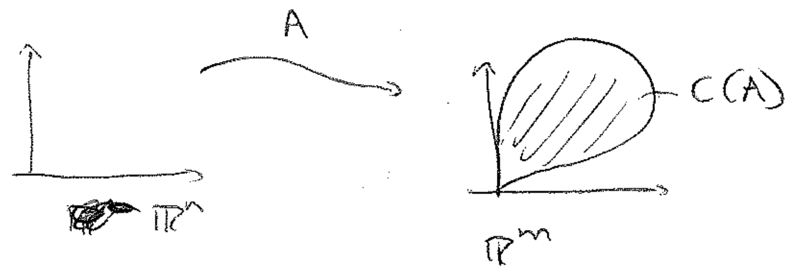
Consider
$$\left.\begin{array}{l} p(x) = x^3 + x^2 \\ g(x) = -x^3 + x \end{array}\right\} \longrightarrow \underbrace{p(x) + g(x)}_{\text{linear combination}} = \underbrace{x^2 + x}_{\text{It is not a polynomial of degree 3 !!!}}$$

II.) The column space.

Def: Column space

The column space of an $m \times n$ matrix A, denoted as $C(A)$, is the collection of all vectors in $\mathbb{R}^m$ that are linear combinations of its columns.

↓



It is the image of the linear transformation given by A.

$C(A)$ is (always) a subspace of $\mathbb{R}^m$ (it might be $\mathbb{R}^m$ itself).

Again, go back to column picture:

$$Ax = b \rightsquigarrow \left[a_1 \middle| a_2 \middle| \cdots \middle| a_n \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = b \rightsquigarrow$$

$$\rightsquigarrow \underbrace{x_1 \begin{bmatrix} a_1 \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_n \end{bmatrix}}_{} = b$$

all the vectors that can be written as a linear combination of the columns form the column space. } Therefore, the system $Ax = b$ has a solution if and only if $b \in C(A)$, or, in different words,

$Ax = b$ has a solution $\Leftrightarrow$ b is a linear combination of the columns of A.

• Example: (Typical)

Given the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ -2 & -1 & -3 \end{bmatrix},$$ what are the equations that define $C(A)$?

(importante)

Sol:

The $C(A)$ are all the vectors that can be written as lin. comb. of the columns of A. That is, all the vectors b that solve $Ax=b$ (for some x) belong to the column space.

So we need to find the conditions (=equations) that a vector b has to satisfy in order to make the system solvable:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 6 & 3 & 9 & b_2 \\ -2 & -1 & -3 & b_3 \end{array}\right] \xrightarrow[R3' = R3 + R1]{R2' = R2 - 3R1} \begin{bmatrix} 2 & 1 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{bmatrix}$$

Thus, $Ax=b$ has solution if

$$\begin{aligned} b_2 - 3b_1 &= 0 \\ b_3 + b_1 &= 0 \end{aligned}$$

The column space is described then by

$$C(A) = \{ b = (b_1, b_2, b_3) \in \mathbb{R}^3 : b_2 - 3b_1 = 0,\ b_3 + b_1 = 0 \}.$$

MATH 312
LECTURE 5

: Column Space and Nullspace of A.

...(From last lecture)

Exercise: Determine if

1) The positive quadrant $(x \geq 0, y \geq 0)$ is a subspace of $\mathbb{R}^2$.

Remark: We will identify the two following spaces as $\mathbb{R}^2$,

$$\text{“}\mathbb{R}^2\ 1\text{”} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ x_1, x_2 \in \mathbb{R} \right\} \text{ with } \begin{cases} \bullet\ \vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}, \\ \bullet\ \lambda \vec{x} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} \end{cases}$$

$$\text{“}\mathbb{R}^2\ 2\text{”} = \left\{ x = (x_1, x_2),\ x_1, x_2 \in \mathbb{R} \right\} \text{ with } \begin{cases} \bullet\ x + y = (x_1 + y_1, x_2 + y_2), \\ \bullet\ \lambda x = (\lambda x_1, \lambda x_2). \end{cases}$$

Solution: It is not a subspace since

$$\underset{(\text{i.e. } 1 \geq 0,\ 1 \geq 0)}{(1,1) \in \text{Positive quadratic}}, \text{ but } -1 \cdot (1,1) = (-1,-1) \text{ does not.}$$

2) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

With the previous definitions, $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since its objects are two-component column vectors while in $\mathbb{R}^3$ are three-component column vectors (so it is not even a subset).

However, under this question, the natural thing is to identify

$\mathbb{R}^2$ with a plane in $\mathbb{R}^3$, i.e., ~~considering~~ to consider that we are talking about (for example)

$$\text{“}\mathbb{R}^2\text{”} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},\ x_1, x_2 \in \mathbb{R} \right\}.$$ This is a subspace of $\mathbb{R}^3$, since

$$\left.\begin{array}{l} \vec{a} \in \text{“}\mathbb{R}^2\text{”} \ (\text{i.e., } a_3 = 0) \\ \vec{b} \in \text{“}\mathbb{R}^2\text{”} \ (\text{i.e., } b_3 = 0) \end{array}\right\} \Rightarrow \alpha\vec{a} + \beta\vec{b} = \begin{bmatrix} \alpha a_1 + \beta b_1 \\ \alpha a_2 + \beta b_2 \\ \alpha 0 + \beta 0 \end{bmatrix} = \begin{bmatrix} \alpha a_1 + \beta b_1 \\ \alpha a_2 + \beta b_2 \\ 0 \end{bmatrix} \in \text{“}\mathbb{R}^2\text{”}$$

3) Polynomials with degree 3 and zero poly.: Not a subspace.

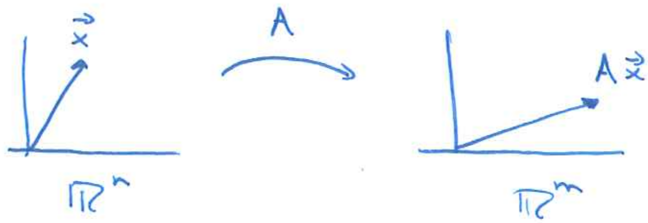
$$\text{Take } \left.\begin{array}{l} p(x) = x^3 + x^2 \\ q(x) = -x^3 + x \end{array}\right\} \Rightarrow p(x) + q(x) = \underbrace{x^2 + x}_{\text{Not degree 3}} (\cancel{\text{so}}.)$$

Easier: ~~the zero vector, i.e., the polynomial $p(x) = 0$~~ is not a 3rd ~~degree polynomial~~ //

II.) The Column Space, $C(A)$ ($\sim$ Sec. 3.1)

Consider a matrix A with m rows and n columns.

That is,

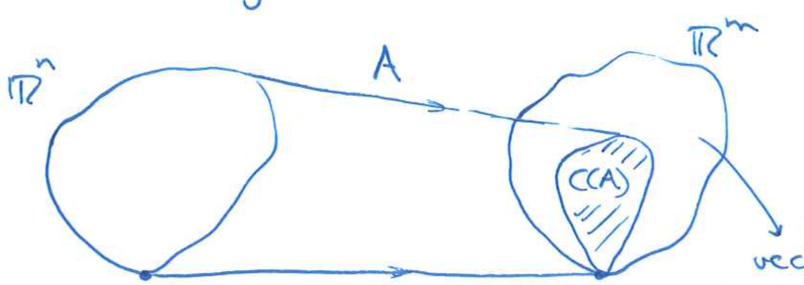


$$\begin{array}{c c} & 3 \\ 2 & \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix}}_{\substack{A \\ (2\times 3)}} \end{array} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\substack{\vec{x} \\ \mathbb{R}^3}} = \underbrace{\begin{bmatrix} x_1 + x_3 \\ 3x_1 - x_2 + x_3 \end{bmatrix}}_{A\vec{x} \quad \mathbb{R}^2}$$

Def: Column Space

The column space of a matrix A $m\times n$ is the collection of <u>all</u> vectors in $\underline{\mathbb{R}^m}$ that might be output by that matrix, i.e., $C(A) = \{\vec{b} \in \mathbb{R}^m : \exists \vec{x} \in \mathbb{R}^n \text{ with } A\vec{x} = \vec{b}\}$.

→ In other words, it is the "image" of the linear function described by A.



All $\mathbb{R}^n$ is sent by A to ~~A~~ $C(A)$ ‖

vectors here can never be reached through A. I.e., $\nexists\, \underset{\mathbb{R}^m}{\underset{\cap}{b}} \,/\, A\underset{x\in\mathbb{R}^n}{x} = b$!!

Remark: $C(A)$ is always a subspace of $\mathbb{R}^m$
(prove it as an exercise).

- Recall the column picture:

$$Ax = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 \begin{bmatrix} a_1 \end{bmatrix} + \dots + x_n \begin{bmatrix} a_n \end{bmatrix}}_{\text{linear combination of the columns of } A.}$$

So,

Def. 2: The column space of A ($m \times n$) consists of all linear combinations of the columns of A.

- Important consequence:

$$\| \; Ax = b \text{ has solution} \iff b \in C(A) \; \|.$$

- How can we describe the column space of a given matrix A?

We just need to find the conditions (equations) on b such that $Ax = b$ has always solutions.

Remark: It is possible that $C(A) = \{\vec{0}\}$ or $C(A) = \mathbb{R}^m$.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}$, $C(A)$?

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 0 & b_2 \\ 2 & 1 & b_3 \end{array}\right] \xrightarrow[R3' = R3 - 2R1]{R2' = R2 - 3R1} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -6 & b_2 - 3b_1 \\ 0 & -3 & b_3 - 2b_1 \end{array}\right] \xrightarrow{R3'' = R3' - \frac{1}{2}R2'}$$

$$\to \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -6 & b_2 - 3b_1 \\ 0 & 0 & b_3 - 2b_1 - \frac{1}{2}(b_2 - 3b_1) \end{array}\right] \Rightarrow b_3 - 2b_1 - \frac{1}{2}(b_2 - 3b_1) = 0 .$$

So, $C(A) = \left\{ \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : b_3 - \frac{1}{2}b_2 - \frac{1}{2}b_1 = 0 \right\}$

Remark: In this case we also "see" very easily that $C(A)$ is the space given by the linear combinations of $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ (which aren't linearly dependent), so $C(A)$ is a plane.

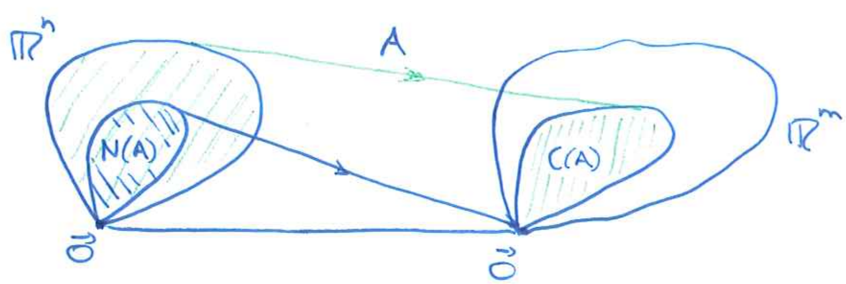
→ We will see later how to find a "basis" for $C(A)$.

III.) The Nullspace, N(A). (~ Sec. 3.2.)

Let A $m \times n$ matrix.

Def: The nullspace of a matrix A $(m \times n)$ is the collection of all vectors in $\mathbb{R}^n$ that are sent to the zero vector in $\mathbb{R}^m$ by A, i.e.,

$$N(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = 0 \}$$



A "kills" N(A)

- How to find N(A)?

Well we know that! Just solve $Ax = 0$!

Example: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$, N(A)?

Solve $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ~> Gaussian Elimination.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[R3' = R3 - 3R1]{R2' = R2 - 2R1} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{bmatrix} \xrightarrow[R3'' = R3' - R2']{} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

So the solution is given by $\left.\begin{aligned} x_1 + 3x_2 + 5x_3 &= 0 \\ -5x_2 - 10x_3 &= 0 \end{aligned}\right\}$

Then x_3 can be seen as a parameter ("free variable") and solve for x_2, x_1: $x_3 = \lambda$,

$$\left.\begin{aligned} &E2 \to \quad x_2 = -2\lambda \\ &E1 \longrightarrow \quad x_1 = -3(-2\lambda) - 5\lambda = \lambda \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

That is,

$$N(A) = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{aligned} x_1 + 3x_2 + 5x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned} \right\},$$

or we can write

$$N(A) = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \lambda \in \mathbb{R} \right\}$$

- If our matrix has too many "free variables", there is a better way to proceed (more systematic):

1) Find the RREF ("Reduced Row Echelon Form") of A, i.e, same steps as in Gauss-Jordan → We continue with Gaussian elimination upwards and finally divide by the pivots.

2) The columns without pivot are "free variable", and the rest of variables are given in terms of the free ones.

• Example: $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$, $N(A)$?

Let's find the RREF of A:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{(change } R3 \Leftrightarrow R4)]{}$$

$$\longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variable

Solution:
$$\left.\begin{aligned} x_2 &= \lambda \\ x_1 &= -2\lambda \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

• **Another) example:** $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 1 & 0 & 5 \\ 3 & 4 & 5 & 5 \\ 1 & -2 & -5 & 5 \end{bmatrix}$, $N(A)$

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 1 & 0 & 5 \\ 3 & 4 & 5 & 5 \\ 1 & -2 & -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & -5 & -10 & 5 \\ 0 & -5 & -10 & 5 \\ 0 & -5 & -10 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & -5 & -10 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & -1 & 3 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Note: The RREF is unique, no matter how we obtain it).

Free variables

Solution: Denote $x_3 = \alpha$, $x_4 = \beta$, then

$$\left.\begin{aligned} x_1 &= \alpha - 3\beta \\ x_2 &= -2\alpha + \beta \\ x_3 &= \alpha \\ x_4 &= \beta \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

• **Remark:** The number of free variables "counts" the "dimension" of the nullspace.

IV| The Complete Solution to $Ax=b$.

Example: $A=\begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & -2 & 0 & 0 \\ -1 & 2 & 1 & 2 \\ -2 & 3 & 1 & 2 \end{bmatrix}$, $b=\begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}$.

Sol: We will perform Gaussian elim. on $[A\,|\,b]$ until we get its RREF:

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 2 & 1 \\ 2 & -2 & 0 & 0 & -2 \\ -1 & 2 & 1 & 2 & 2 \\ -2 & 3 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & -2 & -2 \\ 2 & -2 & 0 & 0 & -2 \\ 0 & 1 & 1 & 2 & 1 \\ -2 & 3 & 1 & 2 & 3 \end{array}\right] \to$$

$$\to \left[\begin{array}{cccc|c} 1 & -2 & -1 & -2 & -2 \\ 0 & 2 & 2 & 4 & 2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -2 & -1 & -2 & -2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & -2 & -1 \end{array}\right] \to$$

$$\to \left[\begin{array}{cccc|c} \boxed{1} & -2 & -1 & -2 & -2 \\ 0 & \boxed{1} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 2 & 0 \\ 0 & \boxed{1} & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Free variables

Thus:

$$\left.\begin{array}{l} x_3=\alpha \\ x_4=\beta \\ x_1=-\alpha-2\beta \\ x_2=-\alpha-2\beta+1 \end{array}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{constant}} + \underbrace{\alpha\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{\text{Nullspace.}}$$

This is a particular solution (any choice of α,β gives different "particular" solutions)

MATH 312 LECTURE 6 : Linear Independence and Basis.

(IV) The Complete Solution to $Ax=b$ ($\sim$ Sec. 3.3))

• <u>Solving $Ax=b$</u> : Summary

1) $\exists$ solution if $b \in C(A)$

2) If $N(A)=\{\vec{0}\}$, at most one solution.

The solution to $Ax=b$ in the general case is given by

$x = x_p$ + "Nullspace".

Indeed, for $x_n \in N(A)$ and x_p a particular solution of $Ax=b$,

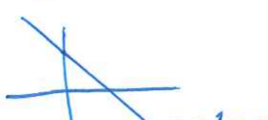
$$A(x_p+x_n) = \underbrace{Ax_p}_{=b} + \underbrace{Ax_n}_{=0} = b$$

Let's see this with an easy example:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{nullspace}}$$

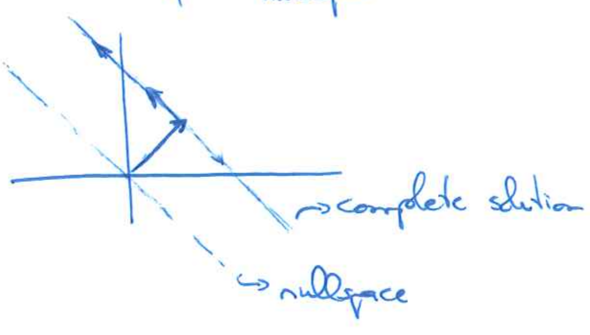
Classical viewpoint:

$x+y=1 \to$ line



New viewpoint:

The Nullspace translated by x_p.



• <u>Possible situations:</u>

Def. Rank of A

The rank of a matrix A is the number of pivots.

Two important cases:

1) Full column rank: $r = n$ ($n^{\underline{o}}$ of pivots $= n^{\underline{o}}$ of columns)

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} \rightsquigarrow \begin{bmatrix} \square & & \\ & \square & \\ & & \square \\ 0 & 0 & 0 \\ & \vdots & \end{bmatrix}$$

RREF of A

→ All columns have pivots ⇒ No free variables ⇒

⇒ $\boxed{N(A) = \{\vec{0}\}}$ ⇒ If there is solution, only one.

There might not be one.

$$\begin{bmatrix} \square & & \\ & \square & \\ & & \square \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vdots \\ \vdots \\ 1 \\ 0 \end{bmatrix}$$ ← This equation $0 = 1$!!

2) Full row rank: $r = m$ ($n^{\underline{o}}$ of pivots $= n^{\underline{o}}$ of rows)

$$\begin{bmatrix} \square & & & \cdots \\ & \square & & \cdots \\ & & \square & \cdots \end{bmatrix}$$

→ All rows have pivots ⇒ $Ax = b$ always has a solution ⇒

⇒ $\boxed{C(A) = \mathbb{R}^m}$

There can be free variables (in particular, there are $n - r = n - m$ free variables).

• In short, 4 situations depending on the rank (≡ n° of pivots):

1) $r = n = m$ (Full rank)

$\begin{bmatrix} \square & & \\ & \square & \\ & & \square \end{bmatrix}$ A is square and invertible

} $Ax=b$ has one and only one solution.

2) $r = n$, $r < m$
(Full column rank)

$\begin{bmatrix} \square & & \\ & \square & \\ & & \square \\ & \text{zeros rows} & \end{bmatrix}$

} $Ax=b$ has 0 or 1 solutions.

3) $r = m$, $r < n$
(Full row rank)

$\begin{bmatrix} \square & & & \vdots & \vdots \\ & \square & & \vdots & \vdots \\ & & \square & \vdots & \vdots \end{bmatrix}$

free variables

} $Ax=b$ has ∞ solutions.

(N(A) contains nonzero vectors).

4) $r < n$, $r < m$

$\begin{bmatrix} \square & & \vdots & \vdots \\ & \square & \vdots & \vdots \\ & \text{zero rows} & & \end{bmatrix}$

free variables

} $Ax=b$ has 0 or ∞ solutions.

VI Linear Independence, basis and dimension (~ Sec. 3.4)

Def: Linear independence.

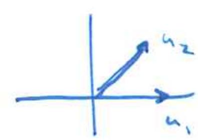
A set of vectors $u_1, u_2, \ldots, u_n$ is called linearly independent if

$$\underset{\text{(vector)}}{0} = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n \Rightarrow \underset{(\in \mathbb{R})}{c_1 = c_2 = \ldots = c_n = 0}.$$

↳ In other case, they are said to be linearly dependent.

Examples:

1) $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let's check:

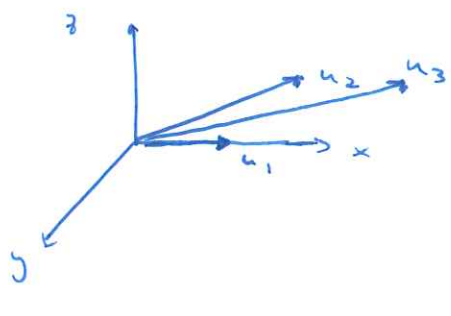


$$c_1 u_1 + c_2 u_2 = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \to c_1 = 0 \\ c_2 = 0 \end{cases}, \text{ so they are l.i.}$$

2) $u_1 = [1 \ 0 \ 0]^T$, $u_2 = [2 \ 0 \ 1]^T$, $u_3 = [3 \ 0 \ 1]^T$

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = \begin{bmatrix} c_1 + 2c_2 + 3c_3 \\ 0 \\ c_2 + c_3 \end{bmatrix} = 0 \Rightarrow \begin{cases} c_1 + 2c_2 + 3c_3 = 0 \\ c_2 + c_3 = 0 \end{cases}$$

There are infinite c_1, c_2, c_3 that satisfy those equations!
Not only $c_1 = c_2 = c_3 = 0 \Rightarrow$ The vectors are dependent.



(They all "live" in the plane $y = 0$")

3) [Method] How to decide if a given set of vectors are l.i. or l.d.?

Ans:

$u_1, \dots, u_n$ are l.i. $\Leftrightarrow$

$\{ c_1 u_1 + \dots + c_n u_n = 0 \Rightarrow c_1 = \dots = c_n = 0 \} \Leftrightarrow$

$\{ \left[u_1 | \dots | u_n \right] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \Rightarrow c_1 = \dots = c_n = 0 \} \Leftrightarrow$

vectors

$\left[u_1 | \dots | u_n \right]$ has zero nullspace $\Leftrightarrow$

the RREF of $\left[u_1 | \dots | u_n \right]$ has no free variables, i.e. all columns have pivots.

So, in conclusion,

Given $u_1, \dots, u_n$, to decide if they are l.i.:

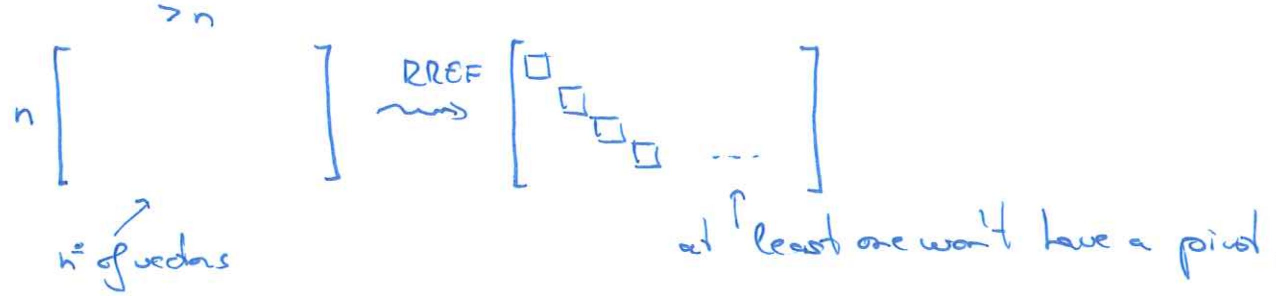
1) Create the matrix $\left[u_1 | \dots | u_n \right]$

2) Eliminate until finding the RREF.

3) Every column has a pivot? $\xrightarrow{\text{Yes}}$ The vectors were l.i.

No $\rightarrow$ The vectors are l.d.

• **Remark**: If there are more than n vectors in $\mathbb{R}^n$, they have to be linearly dependent.



Example: (No column vectors)

1) Show that the matrix

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$ are l.i. (no hecho en clase)

$$c_1 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1+2c_2 & 2c_1+2c_2 \\ 0 & c_1+c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} c_1+2c_2=0 \\ 2c_1+2c_2=0 \\ c_1+c_2=0 \end{array}\right\} \Rightarrow c_1=c_2=0 \text{ (is the only solution).}$$

2) Show that the polynomials $x,\ x^2+2,\ 3x^2,\ 4x+5x^2$ are l.d.

$$c_1 x + c_2(x^2+2) + c_3(3x^2) + c_4(4x+5x^2) =$$

$$= x^2(c_2+3c_3+5c_4) + x(c_1+4c_4) + 2c_2 = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} 2c_2=0 \\ c_1+4c_4=0 \\ c_2+3c_3+5c_4=0 \end{array}\right\} \Rightarrow \begin{array}{l} c_2=0 \\ c_1+4c_4=0 \\ 3c_3+5c_4=0 \end{array} \Rightarrow \text{There are infinite solutions!} \Rightarrow \text{the vectors are l.d.}$$

• <u>Def</u>: Span

Let V be a vector space, and $u_1, \dots, u_n$ a collection of vectors in V.

We say that $u_1, \dots, u_n$ span V if every vector $b \in V$ is a linear combination of $u_1, \dots, u_n$, i.e.,

$\{u_1, \dots, u_n\}$ spans V $\Leftrightarrow$ $\forall\, b \in V$, $\exists\, c_1, \dots, c_n$ scalars such that

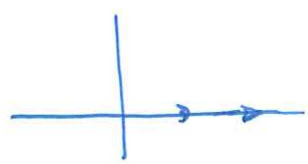
$$b = c_1 u_1 + \dots + c_n u_n .$$

<u>Examples</u>:

1) $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ doesn't span $\mathbb{R}^2$:

Let $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 0 \end{bmatrix} \Rightarrow$

$\Rightarrow \begin{matrix} c_1 + 2c_2 = 0 \\ 1 = 0 \end{matrix}$! no solution!.



2) $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^3$:

Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$. We need to check that there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$b = c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 .$$

That is,

we need to check that the system

$$\begin{array}{cccc} \\ \end{array}\begin{array}{c}\left[\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ u_1 & u_2 & u_3 & u_4 \end{array}\end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$ has solution no matter which b,

that is, we need to check that $C(A) = \mathbb{R}^3$, or, as we have previously seen, that there is a pivot in <u>each row</u> (in other words, there are not rows like $0=1$).

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix} //.$$

3) $u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ spans the vector space $V = \left\{ b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3 : b_1 + b_2 + b_3 = 0 \right\}$.

We need to check that any b with $b_1 + b_2 + b_3 = 0$ can be written as

$b = c_1 u_1 + c_2 u_2$ for some $c_1, c_2 \in \mathbb{R}$.

That is, that

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 - b_2 \end{bmatrix}$$ always has a solution $\longrightarrow$

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ -1 & 1 & b_2 \\ 0 & -1 & -b_1-b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \boxed{1} & 0 & b_1 \\ 0 & \boxed{1} & b_2+b_1 \\ 0 & 0 & 0 \end{array}\right]$$ → There is always a solution //.

• Def: Basis

Let V be a vector space. The the vectors $\{u_1, \dots, u_n\}$ are a basis for V if

1) They are linearly independent. (~ i.e., "not too many")
and
2) They span V. (~ i.e., "not too few").

Remarks:

• In general, there are infinitely many basis for a vector space.

• In $\mathbb{R}^n$, all the basis contains n vectors.

↳ If there are less, they can't span $\mathbb{R}^n$.
↳ If there are more, they can't be l.i.

Exercise: Given a basis $\{u_1, \dots, u_n\}$ of V, any other vector $b \in V$ is written as a linear combination of $u_1, \dots, u_n$ in an unique way.

Solution:

Assume we find two different combinations:

$$\left.\begin{array}{l} b = a_1 u_1 + \dots + a_n u_n \\ b = c_1 u_1 + \dots + c_n u_n \end{array}\right\} \text{Then,}$$

$$b - b = 0 = (a_1 - c_1)u_1 + \dots + (a_n - c_n)u_n \xrightarrow{\text{since } u_1, \dots, u_n \text{ are l.i.}} a_1 - c_1 = 0 = a_2 - c_2 = \dots = a_n - c_n \Rightarrow$$

$$\Rightarrow a_1 = c_1,\ a_2 = c_2,\ \dots,\ a_n = c_n \,/\!/.$$

Example:

1) ~~Find~~ Check that $\{u_1 = [1\ 0\ 0]^T, u_2 = [0\ {-2}\ 0]^T, u_3 = [1\ 1\ 1]^T\}$ is a basis for $\mathbb{R}^3$.

Sol:

Are they l.i. and span $\mathbb{R}^3$?

$$\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & -2 & 1 & b_2 \\ 0 & 0 & 1 & b_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 0 & 0 & b_1 - b_3 \\ 0 & \boxed{1} & 0 & -(b_2 - b_3)/2 \\ 0 & 0 & \boxed{1} & b_3 \end{bmatrix}$$

All columns have pivots $\Rightarrow$ They are l.i.

All rows have pivots $\Rightarrow$ There is solution for all $b \Rightarrow$ They span $\mathbb{R}^3$ //.

2) Find a basis for the subspace given by $y=x$ in $\mathbb{R}^2$.

We are given the subspace $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = y \right\}$.

That is, all vectors that satisfy $x - y = 0$, or, in matrix notation,

$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$. (So we are looking for the nullspace of $[1 \ -1]$!).

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \longrightarrow \left. \begin{matrix} y = \alpha \\ x = y = \alpha \end{matrix} \right\} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

pivot, free variable

Our basis.

3) ~~Find a basis~~ Check that the vectors $p_1(x) = x$, $p_2(x) = x^2+2$, $p_3(x) = x+1$ form a basis of $V = \{$polynomials of degree no more than 2$\}$.

Let's check $\begin{cases} 1) \text{ l.i.} \\ 2) \text{ span.} \end{cases}$

1) $(\alpha p_1 + \beta p_2 + \gamma p_3)(x) = 0 \Rightarrow \alpha = \beta = \gamma = 0$?

$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) = \alpha x + \beta x^2 + 2\beta + \gamma x + \gamma = 0 \Rightarrow$

$$\Rightarrow \begin{cases} \beta = 0 \\ \alpha + \gamma = 0 \\ 2\beta + \gamma = 0 \end{cases} \rightarrow \gamma = 0 \rightarrow \alpha = 0 \Rightarrow \alpha = \beta = \gamma = 0.$$

So they are l.i.

2) Consider any polynomial in V, i.e.,

$b(x) = a + bx + cx^2$.

Can we find c_1, c_2, c_3 such that $b = c_1 p_1 + c_2 p_2 + c_3 p_3$?

$a + bx + cx^2 = c_1 x + c_2 x^2 + 2c_2 + c_3 x + c_3 \Leftrightarrow$

$$\Leftrightarrow \left.\begin{aligned} a &= 2c_2 + c_3 \\ b &= c_1 + c_3 \\ c &= c_2 \end{aligned}\right\} \Leftrightarrow \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 2 & 1 & a \\ 0 & 1 & 0 & c \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & a-2c \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & b-a+2c \\ 0 & \boxed{1} & 0 & c \\ 0 & 0 & \boxed{1} & a-2c \end{array}\right]$$

So yes, we can always find a solution c_1, c_2, c_3. Therefore, p_1, p_2, p_3 span V.

MATH 312
LECTURE 7

: Fundamental Theorem of Linear Algebra. ($\sim$ Sec. 3.5)

Let A be an $m \times n$ matrix. There are "Four Fundamental" Subspaces associated to it:

1) $C(A)$ in $\mathbb{R}^m$ $\equiv$ Column Space $\equiv$ Span of the columns of A.

2) $N(A)$ in $\mathbb{R}^n$ $\equiv$ Nullspace $\equiv$ Vectors x such that $Ax=0$.

3) $C(A^T)$ in $\mathbb{R}^n$ $\equiv$ Row Space $\equiv$ Span of the rows of A.

4) $N(A^T)$ in $\mathbb{R}^m$ $\equiv$ Left Nullspace $\equiv$ Vectors x such that $x^T A = 0^T$.

Def: Dimension of a vector space.
Number of elements in a basis of the vector space.

- How to find basis for $C(A)$, $C(A^T)$, $N(A)$, $N(A^T)$?
What are the dimensions of the Four subspaces?

$\boxed{1}$ We will start with matrices R that are in RREF, for example,

$$R = \begin{bmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{pmatrix} m=4 \\ n=5 \end{pmatrix} \quad \text{Rank of } R \equiv r = 3$$

1) $C(R)$

→ Columns with pivots are l.i.

→ Non-pivot columns can be write as lin. comb. of pivots ones.

↳ Pivot columns span $C(R)$.

So, in conclusion:

A basis of $C(R)$ = "pivot columns of R" $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim C(R) = n^{\circ}$ of pivots $= r = \operatorname{rank}(R) = 3$

2) $N(R)$

It is given by the free variable:

$$\begin{matrix} x_2 = \alpha \\ x_4 = \beta \end{matrix} \rightsquigarrow \left.\begin{matrix} x_1 = -2\alpha - \beta \\ x_3 = -2\beta \\ x_5 = 0 \end{matrix}\right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

comes from the free variables: are l.i.

Form a basis.

Basis of $N(R) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim N(R) = 2 = 5 - 3$ //.

→ || Rank-Nullity Theorem: $\dim N(R) = n - r$ ||

$\left(\dim N(R) + \underset{\overset{|||}{r}}{\dim C(R)} = \underset{\overset{|||}{n}}{\dim(\text{domain } R)} \right)$

3) $C(R^T)$

Note that rows with pivots are l.i., so they form a basis:

Basis for $C(R^T)$ = "rows with pivots" $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

$\| \dim C(R^T) = n^{o} \text{ of pivots} = \text{rank} = \dim C(R) \|$

Q: $\dim(N(R^T))$? Use Rank-Nullity theorem.

4) $N(R^T)$

By definition, $N(R^T) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x^T R = 0^T \right\}$, i.e, those vectors such that

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \overbrace{\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}^{R} = \begin{bmatrix} 0, 0 & 0 & 0 \end{bmatrix}, \text{ i.e.,}$$

$$\begin{bmatrix} x_1 & 2x_1 & x_2 & x_1 + 2x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow \begin{matrix} x_1 = 0, \ x_2 = 0 \\ x_3 = 0 \end{matrix}.$$

So x_4 is "free".

Basis for $N(R^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\dim N(R^T) = 1$

→ In general, we will have as many free variables as zero rows, that is, $m - r$, so

$\dim N(R^T) = m - r$.

<u>Summary</u>

$$\left.\begin{aligned} &\dim C(R) = r = \dim C(R^T) \\ &\dim N(R) = n - r \\ &\dim N(R^T) = m - r \end{aligned}\right\} \text{ (where } R \text{ is } m \times n\text{)}$$

$\boxed{2}$ For a general matrix A $(m \times n)$.

$$A \xrightarrow[\text{operations}]{\text{row}} \underset{\substack{\uparrow \\ E \text{ is invertible.}}}{EA} = R$$

→ obviously, $N(A) = N(R)$ (so same basis):

$$Ax = 0 \underset{\substack{\uparrow \\ E \text{ is invertible.}}}{\iff} EAx = 0 \iff Rx = 0$$

→ Is $C(A) = C(R)$? <u>NO</u> !!

<u>Ex.</u>

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \text{Previous } R = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

But the position of the independent columns <u>is</u> the same:

$$A = \left[a_1 \middle| \cdots \middle| a_n \right] \sim R = \left[\begin{array}{c|c|c} \underset{\underset{r_1}{\shortparallel}}{Ea_1} & \cdots & \underset{\underset{r_2}{\shortparallel}}{Ea_n} \end{array} \right] \text{ (columns of } R)$$

Since E is invertible (i.e., $Ex \neq 0$ for all $x \neq 0$),

if columns of R are l.i., for our example

$$c_1 r_1 + c_3 r_3 + c_5 r_5 = 0,$$

then, multiplying by E,

$$c_1 a_1 + c_3 a_3 + c_5 a_5 = 0.$$

So $\{r_1, r_3, r_5\}$ are l.i. $\Rightarrow$ $\{a_1, a_3, a_5\}$ are l.i.

- In summary,

  $\dim C(A) = r = \dim C(R)$

  A basis for $C(A)$ = "columns <u>of A</u> in the position where pivots will appear",

  i.e. "columns <u>of A</u> in the same position as those of R".

→ $C(A^T)$: This is the span of the rows.

Clearly, row operations don't modify this, so $C(A^T) = C(R^T)$ //.

(↑ EA: E simply does lin. comb. of the rows of A)

→ $N(A^T)$: dimension?

We know $C(A^T) = r$, so by Rank-Nullity theorem,

$\dim N(A^T) = m - r$.

Basis for $N(A^T)$?

We will have

$$\underbrace{\underset{E}{\begin{bmatrix} \text{---}\ E_1\ \text{---} \\ \vdots \\ \text{---}\ E_m\ \text{---} \end{bmatrix}} \underset{A}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}}_{= \begin{bmatrix} E_1 A \\ \vdots \\ E_m A \end{bmatrix}} = \underset{R}{\begin{bmatrix} - & - & & & \\ - & - & & & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

So the last row is $[\text{---}\ E_m\ \text{---}]\, A = [0\ 0\ 0\ 0\ 0]$

↑

this is the left-Nullspace!

Conclusion: A basis of the left Nullspace is given by the $m-r$ rows of the matrix that transform A into R //.

(no hecho en clase)

Q: <u>Example</u>: $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$ Find a basis for $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$, write the (independent) equations that define them and draw together $C(A), N(A^T)$ and $C(A^T), N(A)$

<u>Ans</u>:

We can "see" that columns two and three are multiples of the first one; and row 2 is row 1 times 2, so:

Basis of $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ (subspace of $\mathbb{R}^2$); Eqs: $C(A) = \{ x \in \mathbb{R}^2 : 2x_1 - x_2 = 0 \}$

Basis of $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$ (subspace of $\mathbb{R}^3$); Eqs $C(A^T) = \left\{ x \in \mathbb{R}^3 : \begin{matrix} x_2 - 3x_1 = 0 \\ x_3 - 2x_1 = 0 \end{matrix} \right\}$

Therefore,

$\dim(N(A)) = 3 - 1 = 2$ (of course! There are two free columns),

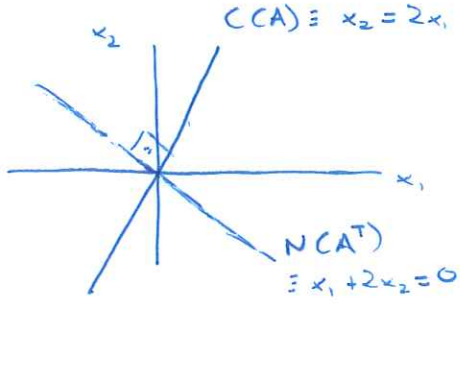
$\dim(N(A^T)) = 2 - 1 = 1$ (" " " is one free row)

Basis of $N(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$; Eqs: $N(A) = \{ x \in \mathbb{R}^3 : x_1 + 3x_2 + 2x_3 = 0 \}$

Basis of $N(A^T) = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$; Eqs: $N(A^T) = \{ x \in \mathbb{R}^2 : x_1 + 2x_2 = 0 \}$

→ We could have done the systematic ~~produce~~ procedure:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \xrightarrow{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \boxed{1} & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{R}$$



How does a subspace look like? (no hecho en clase)

In $\mathbb{R}^n$, we know they are lines, planes, hyperplanes...

Example: Let

$V = \{$polynomials of degree no more than $2\} = \{p$ pol. $: p(x) = a + bx + cx^2\}$

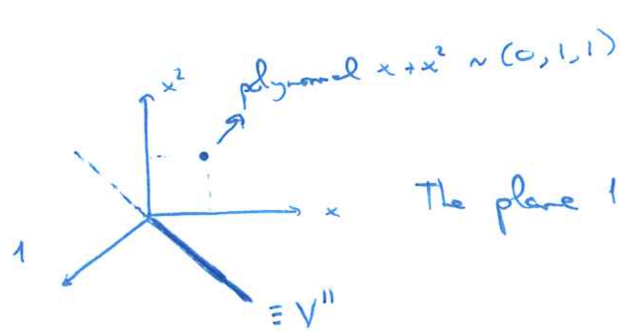
$V' = \{p$ pol. $: p(x) = a + bx\}$, $V'' = \{p$ pol: $p(x) = a + ax\}$.

We can find a basis for V: $\{1, x, x^2\}$

→ of course it spans V

→ Check that they are l.i.

So now we can "draw":



The plane 1-x is V'.

Any polynomial is now seen as a "point" or "vector":

$$p(x) = a + bx + cx^2 \rightsquigarrow (a,b,c)$$

• Moreover, we can think of linear transformations. For example, a derivative:

$$p(x) = a + bx + cx^2 \rightsquigarrow p'(x) = b + 2cx$$

In matrix form,

(no visto en clase)

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{D}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$$

polynomial $a + bx + cx^2$ — derivative $p'(x) = b + 2cx$

(We are using the basis $\{1, x, x^2\}$)

What is $C(D)$ and $N(D)$?

- Free variable → First column, so,

$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the nullspace if $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, that is,

$p(x) = \alpha \equiv$ constant polynomials. Of course! The derivative makes all constants equal to zero.

- Column space ⇝ Basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$, that is, all possible combinations have third component equal to zero, i.e., they are polynomials of degree 1. Of course! The derivative always produces polynomials with one degree less.