Some more problems (Chapt. 2/3)

1 True or false (from past exams)

a) The collection of invertible 4×4 matrices form a subspace of the collection of all 4×4 matrices.

b) If matrices satisfy $AB = I$, then A is invertible and $A^{-1} = B$.

c) If both $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ have solutions, then so does $Ax = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

d) The set of polynomials with degree less or equal than 2 is a subspace of the polynomials.

e) The set of points $y = x + 1$ (a line) is a subspace of the plane.

f) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

g) To multiply matrices AB, if A has n rows then B must have n columns.

h) The unsymmetric 3×3 matrices form a subspace of the 3×3 matrices.

i) If A admits an LU decomposition (lower/upper triangular), then $A^{-1} = U^{-1}L^{-1}$.

<u>Solutions</u>:

a) False. The zero matrix, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, is not invertible.

• Remark: We assume that the statement refers to the vector space formed by the 4×4 matrices with the usual addition and scaling.

• In general, we would need to check that:

Given any A, B 4×4 invertible matrices, then $\alpha A + \beta B$ is also a 4×4 invertible matrix, for any $\alpha, \beta \in \mathbb{R}$.

It is clear that we can always choose $\alpha = 1$, $\beta = -1$, $B = A$.

b) False. Counterexample: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

c) True. Proof: (linearity)

Denote x_1 the solution to $Ax_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and x_2 the one to $Ax_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Notice that $$\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}1\\1\\1\end{bmatrix} = Ax_1 - 2Ax_2 = A(x_1 - 2x_2).$$

So the system $Ax = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ has the solution ~~x~~ $x_1 - 2x_2$.

d) True. Proof:

Let $p(x), g(x)$ polynomials with degree no more than 2.

Then, $\alpha p(x) + \beta g(x)$ is again a polynomial with degree no more than 2, for any $\alpha, \beta \in \mathbb{R}$.

e) False. The zero element $(0,0)$ doesn't belong to that line.

Π <u>Remark</u>: We assume the statement refer to the following vector space when it says "the plane":

$V = \{(x,y) : x, y \in \mathbb{R}\}$ with 1) Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$. 2) Scaling: $\lambda(x,y) = (\lambda x, \lambda y)$

Notice that this space "is not" $\mathbb{R}^2$ in the way we have defined it in class (as column vectors). However, we can trivially identify both spaces. We won't in general distinguish them, so we will say (with abuse of terminology) that $y = x+1$ is not a subspace of $\mathbb{R}^2$.

In other words, we will identify points (x, y) and column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$. ✓

j) "False".

As defined in class, $\mathbb{R}^3 = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ with } u_1, u_2, u_3 \in \mathbb{R} \right\}$ with the usual addition and scaling. On the other hand,

$\mathbb{R}^2 = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ with } u_1, u_2 \in \mathbb{R} \right\}$ (usual addition and scaling).

The objects in $\mathbb{R}^2$ are different than those in $\mathbb{R}^3$ (two components column vectors versus three component ones), so $\mathbb{R}^2$ is not a subcollection of $\mathbb{R}^3$.

- <u>But</u>: With abuse of terminology, we can identity the space $\mathbb{R}^2$ with the following subspace of $\mathbb{R}^3$:

$$V = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}, \text{ with } u_1, u_2 \in \mathbb{R} \right\}.$$

Notice that we can identify $\mathbb{R}^2$ with any plane in $\mathbb{R}^3$ that goes through the origin (and these are subspaces of $\mathbb{R}^3$).

g) False. Counterexample: $\left.\begin{array}{l} A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \text{ (1 row)} \\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ (3 columns)} \end{array}\right\} AB = 0.$

h) False. ~~Count~~

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (both unsymmetric).

We can find a linear combination of A, B that <u>is not</u> unsymmetric:

$$A + B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is not unsymmetric}$$

i) False. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{U}$, but

A doesn't even have an inverse.

<u>Remark</u>: If A has an inverse, then it would be true.

$\boxed{2}$ Consider the following subsets of $\mathbb{R}^3$. Which of them are subspaces?

a) $V_1 = \{(x, y, z) : x + y + z = 0\}$

b) $V_2 = \{(x, y, z) : x + y + z^2 = 0\}$

c) $V_3 = \{(x, y, z) : x, y, z \in \mathbb{Q}\}$ ($\mathbb{Q} \equiv$ rational numbers).

d) $V_4 = \{(x, y, z) : xyz = 0\}$

e) $V_5 = \{(x, y, z) : z = 0\}$

Solution:

Remark: We consider the statement refers to the vector space $\mathbb{R}^3$ defined as points (x, y, z) (with usual add. and scaling).

a) V_1 is a subspace:

Let $\left.\begin{array}{l}(x_1, y_1, z_1) \in V_1 \\ (x_2, y_2, z_2) \in V_2\end{array}\right\}$ Then, $\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) \in V_1$? $(\alpha, \beta \in \mathbb{R})$

Let's check it:

$$\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \rightarrow$$

$$(\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2) = 0 \iff$$

$$\iff \alpha\underbrace{(x_1 + y_1 + z_1)}_{=0} + \beta\underbrace{(x_2 + y_2 + z_2)}_{=0} = 0 \checkmark.$$

b) V_2 is not a subspace.

We can find two vectors in V_2 and a linear combination of them which is not in V_2:

$(-1,0,1) \in V_2$, i.e., $-1+0+1^2=0$,

$(-4,0,2) \in V_2$, i.e., $-4+0+2^2=0$. However,

$(-1,0,1)+(-4,0,2)=(-5,0,3) \notin V_2$, since $-5+0+3^2=4 \neq 0$.

c) It depends on what we consider "scalars".

The addition is naturally defined as

$(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2,\, y_1+y_2,\, z_1+z_2)$.

Now, the scaling is also natural: $\lambda(x,y,z)=(\lambda x,\lambda y,\lambda z)$, but we can consider $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{Q}$.

1) $\underline{\lambda \in \mathbb{R}}$: The V_3 is not a subspace, since

$(1,1,1) \in V_3 \longrightarrow$ But $\pi(1,1,1)=(\pi,\pi,\pi) \notin V_3$.

2) $\underline{\lambda \in \mathbb{Q}}$: The V_3 is a subspace, since

$$\lambda_1(x_1,y_1,z_1)+\lambda_2(x_2,y_2,z_2)=(\underbrace{\lambda_1 x_1+\lambda_2 x_2}_{\in\mathbb{Q}},\, \underbrace{\lambda_1 y_1+\lambda_2 y_2}_{\in\mathbb{Q}},\, \underbrace{\lambda_1 z_1+\lambda_2 z_2}_{\in\mathbb{Q}})$$

Remark: One should check that both options verify the

properties of addition / scaling in vector spaces (page 131 Book).
They do because both $\mathbb{R}$ and $\mathbb{Q}$ are "fields".

<u>Remark 2</u>: We just expect from you answer 1).

d) V_4 is not a subspace.

$$\left.\begin{array}{l}(1,0,0)\in V_4\\(0,1,1)\in V_4\end{array}\right\}\ (1,0,0)+(0,1,1)=(1,1,1)\notin V_4 \quad (1\cdot 1\cdot 1=1\neq 0).$$

$$\uparrow$$

$$\left(\begin{array}{l}1\cdot 0\cdot 0=0,\\ 0\cdot 1\cdot 1=0\end{array}\right)$$

e) V_5 is a subspace (one that we can identify with $\mathbb{R}^2$ indeed).

$$\alpha\underbrace{(x_1,y_1,0)}_{\in V_5}+\beta\underbrace{(x_2,y_2,0)}_{\in V_5}=(\alpha x_1+\beta x_2,\ \alpha y_1+\beta y_2,\ \underset{\smile}{0})\in V_5.$$

$$(\alpha,\beta\in\mathbb{R})$$

$\boxed{3}$ Describe the column space and the nullspace of A,

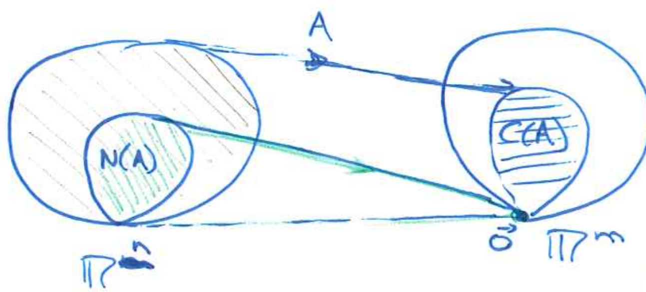
$$A = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix}.$$

Solution:

- Column space of A, C(A): The collection of all vectors that are linear combinations of the columns of A.

- Nullspace of A, N(A): The collection of all vectors that are mapped to the zero-vector.

In general, for A $m \times n$ matrix, C(A) is a subspace of $\mathbb{R}^m$ and N(A) is a subspace of $\mathbb{R}^n$:



A "Kills" the subset N(A).

A sends all $\mathbb{R}^n$ to C(A).

(i.e. C(A) is the image of the linear transformation given by A).

1) Column space of A.

Since $C(A)$ is given by all the vectors that can be written as linear combinations of the columns of A, we are looking for those vectors $\vec{b}$ that make the system $A\vec{x} = \vec{b}$ <u>always</u> solvable.

(Recall that $A\vec{x}$ can be seen as a linear combination of the columns of A).

In conclusion, we want to find the conditions on $b_1, b_2, b_3 \in \mathbb{R}$ such that

$$\begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$ has a solution.

Therefore,

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 6 & 3 & 9 & b_2 \\ 4 & 2 & 6 & b_3 \end{array}\right] \xrightarrow[R3' = R3 - 2R1]{R2' = R2 - 3R1} \left[\begin{array}{cccc} 2 & 1 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 \end{array}\right]$$

has solution if $b_2 - 3b_1 = 0$, $b_3 - 2b_1$. That is, the equations

$$\left.\begin{array}{l} b_2 - 3b_1 = 0 \\ b_3 - 2b_1 = 0 \end{array}\right\} \text{ define the subspace } C(A).$$

2) Nullspace of A.

We look for those vectors $\vec{x}$ that are mapped to zero, i.e.,

$$A\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By solving the system we find the nullspace.

$$\left[\begin{array}{cccc} 2 & 1 & 3 & 0 \\ 6 & 3 & 9 & 0 \\ 4 & 2 & 6 & 0 \end{array}\right] \xrightarrow[R3' = R3 - 2R_1]{R2' = R2 - 3R_1} \left[\begin{array}{cccc} 2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$\Rightarrow$ The system is reduced to $2x_1 + x_2 + 3x_3 = 0$.

So for $\vec{x}$ to be in $N(A)$, it needs to verify that equation.

- In summary, $C(A) = \left\{ \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : b_2 - 3b_1 = 0,\ b_3 - 2b_1 = 0 \right\}$,

$$N(A) = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : 2x_1 + x_2 + 3x_3 = 0 \right\}.$$

4 The LU decomposition of A is given as follows:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

a) Find A.

b) Write down the sequence of operations that takes A to U when performing Gaussian elimination.

c) Solve $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as two triangular systems.

d) Compute U^{-1} by Gauss-Jordan method.

e) Compute A^{-1} if it exists.

Solutions:

a) $$A = LU = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -6 & 0 \\ -3 & -10 & 0 \end{bmatrix}.$$

b) Looking at L directly:

$R2' = R2 + 2R_1$, $R3' = R3 + 3R_1$, $R3'' = R3' - R2'$.

c) $A = LU \to LUx = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. We solve in order the following two systems:

1) $Ly = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to find y:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{array}{l} \to y_1 = 1 \\ \to y_2 = 2 + 2 \cdot 1 = 4 \\ \to y_3 = 3 + 3 \cdot 1 - 4 = 2 \end{array}$$

So $y = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

2) $Ux = y$ to find x:

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{array}{l} \to x_1 = 1 - 4 \cdot 0 - 3\frac{2}{3} = -1 \\ \to x_2 = \frac{1}{2}\left(4 - 6\frac{2}{3}\right) = 0 \\ \to x_3 = 2/3 \end{array}$$

So $x = \begin{bmatrix} -1 \\ 0 \\ 2/3 \end{bmatrix}$ //.

d) $$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R2' = R2 - 2R3]{R_1' = R_1 - 2R2} \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 1 & -2 & 0 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_1'' = R_1' + 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 1/2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$$U^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}$$

e) Since L is always invertible (it is the product of elementary matrices) and U is invertible from d), then

$A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$. What is L^{-1}?

$$L^{-1} = E_{32}E_{31}E_{21} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{R3'' = R3' - R2'} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}_{R3' = R3 + 3R_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R2' = R2 + 2R_1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

So,

$$A^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -3 \\ 0 & -3/2 & -1 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$