Solutions to Midterm 1 (Version A)

$\boxed{P1}$

a) Find $A = LU$

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 0 & -1 & 3 \\ 1 & 2 & -1 & 1 \end{bmatrix} \xrightarrow[R3' = R3 - R1]{R2' = R2 - R1} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \xrightarrow{R3'' = R3 + R2} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$ (we just place the multipliers in their position).

Remark: • You can do the product of the elementary matrices, but takes time.

• Very important: L is always $\boxed{\text{square}}$. Moreover, it is invertible.

• Check that $LU = A$ to avoid errors.

b) From R we see that the first and second column are l.i. (there are pivots). Since $A = E^{-1}R$ the 1^{st} and 2^{nd} columns of A are also l.i. (and span $C(A)$ since the others are l.d.).

So

$$C(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$ "= columns of A in the position of pivots".

· <u>Remarks</u>: · Don't waste time obtaining back A using E^{-1}, that's why you are giving A, E, R (it is good that you know how to do it though).

c) $N(A) = N(R)$, and we can see that there are two free variables

$$R = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_3 = \alpha \\ x_4 = \beta \end{matrix} \left\{ \begin{matrix} x_1 = \alpha - 3\beta \\ x_2 = \beta \end{matrix} \right. \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

pivots free

$$\text{Basis of } N(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

d) We solve $Rx = Eb$:

$$Eb = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \text{ so}$$

$$\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \text{Last equation is } 0 = 2, \text{ so there is no solution.}$$

<u>Remark</u>: The hint won't be given (normally).

$\boxed{P2}$

a) Standard answer/procedure:

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & -2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 1 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 & 2 \\ 0 & 0 & \boxed{2} & 0 & 5 \end{bmatrix}$$

There are three pivots, so three independent vectors.

Therefore, $\dim(V) = 3$.

Remark:
- Since clearly $u_4 = 2u_1$, you can save time removing it.
- Since we are in $\mathbb{R}^3$ (vectors with three components), you can check for three of them and see if you are lucky (in this case you have to explain something).
- Unbelievable answers: $\dim(V) = 4$, $\dim(V) = 5$!!.

b)

$$u_5 = c_1u_1 + c_2u_2 + c_3u_3 = c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}2\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\2\\1\end{bmatrix}$$ and solve for c_1, c_2, c_3.

We know there is a unique solution since u_1, u_2, u_3 are l.i. as we showed in part a).

$\boxed{P3}$

a) V is a <u>line in $\mathbb{R}^2$</u> : $x-y=0$.

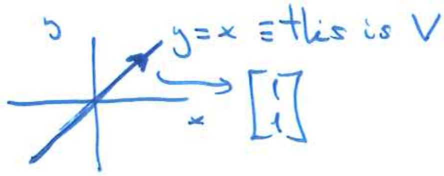
So it has dimension 1, that is, there can only be one vector in the basis(!).

We simply choose one vector satisfying the equation, for example $\begin{bmatrix}1\\1\end{bmatrix}$: basis of $V=\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

• <u>Remark</u>: • You can also do it as finding the basis of a nullspace,

$$x-y=0 \Leftrightarrow \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=0$$

• In this case it should be extremely clear. You can even think about this geometrically:



• Really annoying answers: more than one vector in the basis.

b) You can use the general formula $P = A(A^TA)^{-1}A^T$,

where $A = \begin{bmatrix}1\\1\end{bmatrix}$, but be careful with the notation

(column and row vectors). In this case:

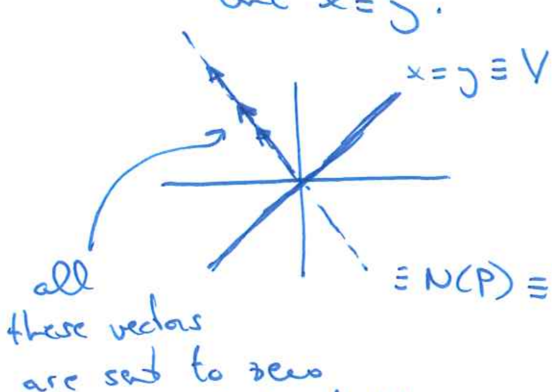
$$P = \begin{bmatrix}1\\1\end{bmatrix}\left(\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} 2^{-1} \begin{bmatrix}1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}.$$

- Remarks:
  - Nicer: Projection onto a line spanned by $\vec{a}$:

$$P = \frac{aa^T}{a^Ta} = \frac{aa^T}{\|a\|^2} = \frac{\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}}{2} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

  - Important: size of the matrix. If it is a projection matrix in $\mathbb{R}^2$, it is going to be applied to vectors in $\mathbb{R}^2$ to produce vectors in $\mathbb{R}^2$, so it has to be 2×2.

c) A line through the origin orthogonal to the given line $x=y$.



$\equiv N(P) \equiv V^\perp$ orthogonal complement of $V \equiv \underbrace{y=-x}$

not asked, but should be very clear in this particular problem.

P4

a) Given an $m \times n$ matrix, like $m\left[\ \overset{n}{A}\ \right]$, we can define a linear application as follows:

$$f_A : \mathbb{R}^n \to \mathbb{R}^m$$
$$\vec{x} \mapsto f_A(\vec{x}) = A\vec{x}$$

That is, we take a vector in $\mathbb{R}^n$, we multiply by A (i.e., we "apply" A) and obtain a new vector Ax in $\mathbb{R}^m$:

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m \qquad m\left[\ \overset{n}{A}\ \right]\left[\overset{n}{x}\right] = \left[\, b \,\right]m$$

(sizes have to match).

(All this wasn't needed)

We know that $C(A)$ is a subspace of $\mathbb{R}^m$ (the "output") and $N(A)$ is a subspace of $\mathbb{R}^n$ (the "input").

Since the vectors in the basis have three components (that is, they are vectors in $\mathbb{R}^3$), it is clear that

$$n = 3$$
$$m = 3$$

• Remarks: • I can see why some people answer $m=2$ (and indeed we will see soon that it makes sense, in some way),

but I ~~really~~ truly don't see how to get

$n=2, m=3$, or $n=6, m=6$! etc.

b) The dimension of a vector space is the number of elements in (any/all) its basis, so

$\dim N(A) = 1$
$\dim C(A) = 2$.

Remark: Don't write $N(A)=1$, $C(A)=2$.

c) $\dim C(A^T) = \text{rank} = \dim C(A) = 2$.

Also, using rank-nullity theorem or fundamental theorem of linear algebra.

d) We know that $C(A^T) = N(A)^\perp$, that is, a vector $x \in \mathbb{R}^3$ is in $C(A^T)$ if is orthogonal to all vectors in $N(A)$. Since $N(A)$ only has one vector in its basis, $x \in \mathbb{R}^3$ if

$$x \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 0 \iff 3x_1 + x_2 + 2x_3 = 0$$

that is, $C(A^T) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : 3x_1 + x_2 + 2x_3 = 0 \right\}$.

Remark: • This was the fastest answer, you didn't need to compute a basis for $C(A^T)$.

• You can also obtain first A and then find the row space, but should be able to do without it.

• Important:

If you answered $\dim C(A^T) = 2$ in c), please check that you give a basis with 2 elements!..
(and analogously, only one equation!)

All this is a plane in $\mathbb{R}^3$.

These things should be totally clear (they are from 240 and even before that).

e) We first place the two vectors in the basis of $C(A)$ to ensure that:

$$A = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 1 & 0 & c \end{bmatrix}, \text{ where } a, b, c \text{ to determine.}$$

We impose that $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ is in $N(A)$:

$$\underbrace{\begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 1 & 0 & c \end{bmatrix}}_{A} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3+2a \\ 4+2b \\ 3+2c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} a = -3/2 \\ b = -2 \\ c = -3/2 \end{array}.$$

So $A = \begin{bmatrix} 1 & 0 & -3/2 \\ 1 & 1 & -2 \\ 1 & 0 & -3/2 \end{bmatrix}$ (Check that $A \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to avoid errors).

$\boxed{P5}$

1) A^TA and AA^T are symmetric : True.

A matrix M is symmetric if $M^T = M$. Let's check it :

$(A^TA)^T = A^T(A^T)^T = A^TA$ ✓

$(AA^T)^T = (A^T)^TA^T = AA^T$ ✓

2) True : Let's denote by V the subspace given by the points on the line. Then we know that $\dim V = 1$.
Since it holds that $\dim V + \dim V^T = 2$, we obtain that $\dim V^T = 1$, so it is a line.

→ <u>Important</u>: It is key the word "In $\mathbb{R}^2$".

3) True :

$$\begin{array}{cc} & 5 \\ 3 & \begin{bmatrix} \square & & & \\ & \square & & \\ & & \square & \dots \end{bmatrix} \end{array}$$

Since there are three pivots, there are 3 independent columns, so they span $\mathbb{R}^3$. So for any b there is a solution at least. Since there are two free variables, there are infinite solutions.

4) True : By contradiction : Assume there exists A^{-1}. Then,

$$A^{-1}Ax = A^{-1}0 = 0 \Rightarrow x = 0 \rightarrow\leftarrow.$$

(there are other good answers given)

5) True: If A is invertible, then A^T is invertible.

(since it implies that A is square of full rank, and A and A^T have the same number of pivots)

So, $A(A^TA)^{-1}A^T = AA^{-1}(A^T)^{-1}A^T = II = I$.

→ key words: "If A is invertible".
When we do projections, A is almost never invertible. It is usually rectangular, since we project onto a subspace of lower dimension.

6) False: $P_{12}^3 = P_{12}\,P_{12}^2 = P_{12}\,I = P_{12}$ //.

(P_{12} → changes rows 1 and 2. So $P_{12}P_{12}$ undoes this: we obtain the identity)

7) $\boxed{\text{TRUE}}$: This is a LINE in $\mathbb{R}^3$!! (Come on!)
(passing through zero)

• Also: $\left.\begin{array}{l} y = 3x+z \\ z+x=0 \end{array}\right\} \equiv \underbrace{\begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ This describes the nullspace of the matrix A.

• Also:
Let (x_1, y_1, z_1) be in V, i.e. $\begin{cases} y_1 = 3x_1 + z_1 \\ z_1 + x_1 = 0 \end{cases}$

(x_2, y_2, z_2) be in V, i.e. $\begin{cases} y_2 = 3x_2 + z_2 \\ z_2 + x_2 = 0 \end{cases}$

We have to check that 1) $(x_1, y_1, z_1) + (x_2, y_2, z_2)$ is in V.
and 2) $\lambda(x_1, y_1, z_1)$ is in V

1) Yes: $\begin{cases} y_1 = 3x_1 + z_1 \\ z_1 + x_1 = 0 \end{cases} \longrightarrow (y_1 + y_2) = 3x_1 + 3x_2 + (z_1 + z_2) = 3(x_1 + x_2) + (z_1 + z_2)$ //

$\begin{cases} y_2 = 3x_2 + z_2 \\ z_2 + x_2 = 0 \end{cases} \longrightarrow (z_1 + z_2) + (x_1 + x_2) = 0$ //

2) Yes: $\begin{rcases} y_1 = 3x_1 + z_1 \\ z_1 + x_1 = 0 \end{rcases} \rightsquigarrow \begin{rcases} (\lambda y_1) = 3(\lambda x_1) + (\lambda z_1) \\ (\lambda z_1) + (\lambda x_1) = 0 \end{rcases}$ //

- Also: $\begin{rcases} y = 3x + z \\ z + x = 0 \end{rcases}$ line with basis $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

So $V = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \lambda \in \mathbb{R} \right\}$, and check:

1) $\vec{u}, \vec{v} \in V \overset{?}{\rightsquigarrow} \vec{u} + \vec{v} \in V$:

$\vec{u} = \lambda_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \vec{v} = \lambda_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \rightsquigarrow \vec{u} + \vec{v} = (\lambda_1 + \lambda_2) \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ ✓.

2) $\vec{u} \in V \overset{?}{\rightsquigarrow} \alpha \vec{u} \in V$:

$\vec{u} = \lambda \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \rightsquigarrow \alpha \vec{u} = (\alpha \lambda) \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ ✓.

8) False:

Take $(1,-1)$: it is in the set since $1^2-1=0$.

However, $2(1,-1)=(2,-2)$ is not in the set: $2^2-2=2\neq 0$.

So it is not closed under scaling.

Not linear

(You should suspect it when you see $x^2+y=0$).

→ It says "set of points...", so we are not talking about polynomials ~~f~~ or functions!.