Solutions to Practice Exam

$\boxed{P1}$

a) $$\begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 2 & 6 & 4 & 4 & 3 \\ 0 & 0 & 2 & 2 & 3 \end{bmatrix} \xrightarrow{R3' = R3 - 2R1} \begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 & 3 \end{bmatrix} \xrightarrow[R4' = R4 + 2R2]{R3'' = R3' + 2R2}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[R1' = R1 - R3'']{R4'' = R4' - 3R3''} \begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R2' = -R2]{R1'' = R1' + R2}$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

b)

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 6 & -4 & -3 & 1 \end{bmatrix}$$ (Check $EA = R$).

c) Denote $b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$. Then $Ax = b$ is equivalent to $EAx = Eb \to$

$$\to \begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 6 & -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Free variables, $\left. \begin{array}{l} x_2 = \alpha \\ x_4 = \beta \end{array} \right\}$ $\begin{array}{l} x_1 = -1 - 3\alpha \\ x_3 = -\beta \\ x_5 = 1 \end{array}$ $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

d)

Basis for $N(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ (subspace of $\mathbb{R}^5$) ("free variables")

Basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$ (subspace of $\mathbb{R}^4$) (pivot columns of A.)

Basis for $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (subspace of $\mathbb{R}^5$) (pivot rows)

Basis for $N(A^T) = \left\{ \begin{bmatrix} 6 \\ -4 \\ -3 \\ 1 \end{bmatrix} \right\}$ (subspace of $\mathbb{R}^4$) (last row of E)

$\boxed{P2}$ a) $2\left[\overset{4}{A}\right] \rightsquigarrow$ A goes from $\mathbb{R}^4$ to $\mathbb{R}^2$ $(m=2)$.

b) $\dim N(A) = 2$.

c) Rank-Nullity Theorem: $\dim N(A) + \dim C(A) = 4 \Rightarrow$

$\Rightarrow \dim C(A) = 2$.

And we also know from a) that $C(A)$ is in $\mathbb{R}^2$!

So $C(A) = \mathbb{R}^2$.

d) $\dim C(A) = 2 \rightsquigarrow 2$ pivot columns.

So R is the following matrix

Nullspace $\rightarrow \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ (from the free variables)

$$R = \begin{bmatrix} 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

pivot columns: 1st and 3rd.

$\boxed{P3}$

a) $C(A)$ is the span of the columns of A. To find a basis we have to select the l.i. columns:

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 3 & 1 \end{bmatrix} \to$$

$$\to \begin{bmatrix} \boxed{1} & 2 & 1 & 2 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & 0 & 0 & \boxed{7} & 3 \end{bmatrix}$$

$$\text{Basis for } C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 7 \end{bmatrix} \right\}.$$

<u>Remark</u>: Since $C(A)$ is in $\mathbb{R}^4$ and $\dim C(A) = 4$, it is clear that $C(A) = \mathbb{R}^4$, so a different and more convenient basis is the canonical one:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b)

Since $C(A)=\mathbb{R}^4$, it is trivial that b is $\in C(A)$

(any b is!).

Choosing the canonical basis,

$$b=\begin{bmatrix}2\\0\\1\\8\end{bmatrix}=2\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+1\cdot\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+8\begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Remark: If we haven't noticed this, we should do:

$$b=\begin{bmatrix}2\\0\\1\\8\end{bmatrix}=a\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+b\begin{bmatrix}2\\1\\0\\0\end{bmatrix}+c\begin{bmatrix}2\\2\\1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\\2\\7\end{bmatrix}=$$

$$=\begin{bmatrix}1&2&2&0\\0&1&2&1\\0&0&1&2\\0&0&0&7\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$$ and solve the system. (Longer but not that much since it is in triangular form).

$\boxed{P4}$

a) $2x - y + z = 0 \rightarrow \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$$y = \alpha,\ z = \beta \rightarrow x = \frac{1}{2}(\alpha - \beta) \rightarrow$$

$$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

Basis for the plane $= \left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

b) Construct $A = \begin{bmatrix} 1/2 & -1/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$$P = A(A^TA)^{-1}A^T$$

$$A^TA = \begin{bmatrix} 1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5/4 & -1/4 \\ -1/4 & 5/4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \rightarrow$$

$$\rightarrow (A^TA)^{-1} = \frac{1}{6} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$P = \frac{1}{6} \begin{bmatrix} 1/2 & -1/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \frac{1}{2} = \frac{1}{24} \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 10 & 2 \\ -4 & 2 & 10 \end{bmatrix} =$$

$$= \frac{1}{24} \begin{bmatrix} 8 & 8 & -8 \\ 8 & 20 & 4 \\ -8 & 4 & 20 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & 1 \\ -2 & 1 & 5 \end{bmatrix} =$$

(symmetric!)

$\boxed{PS}$

1) <u>False</u>. Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2) If $N(A) = \{0\}$ then A has no free variables, so it has full column rank. Since it is square, it is also full row rank. So <u>True</u>.

3) <u>False</u>: such a matrix cannot have an inverse, since the two first rows are lin. dependent.

Ex: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$!!

4) The nullspace of the projection onto a line is its orthogonal complement. In $\mathbb{R}^3$, that cannot be a line since their dimensions don't add up to 3. So <u>False</u>.

5) <u>False</u>:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = A^T \rightsquigarrow AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

6) $\begin{array}{c|c} & 5 \\ 3 & \begin{bmatrix} \square & & \\ & \square & \\ & & \square \end{bmatrix} \end{array}$ Its rows are l.i., but they are only 3, so they cannot span $\mathbb{R}^5$. So. <u>False</u>.

7) <u>False</u>.

Take $(1,3)$ in that set $(3 = 3 \cdot 1^2)$.

But $2(1,3) = (2,6)$ doesn't verify it: $6 \neq 3 \cdot 2^2$ //.

8) <u>True</u>:

Let p_1, p_2 in the set: $p_1(x) = a_1 x^2$
$p_2(x) = a_2 x^2$

Then, 1) Scaling: $(\lambda p_1)(x) = \lambda p_1(x) = (\lambda a_1) x^2$ ✓.

2) Addition: $(p_1 + p_2)(x) = p_1(x) + p_2(x) = a_1 x^2 + a_2 x^2 = (a_1 + a_2) x^2$. ✓