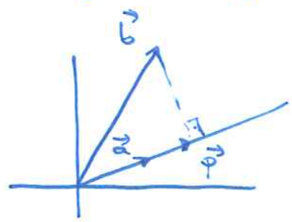
MATH 312 LECTURE 11 : Orthogonal basis and Gram-Schmidt. (~ Sec. 4.4)

(From last days...)

- Projection of a vector $\vec{b}$ onto a line:



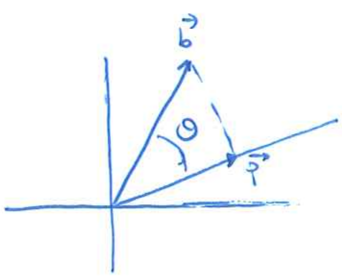
1) We choose $\vec{a}$ (basis for the line)

2) Then, $\vec{p} = \dfrac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}$

<u>Remark</u>: We <u>choose</u> $\vec{a}$. So we can always choose it such that $\|\vec{a}\| = 1$.

In this case, $\vec{p} = (\vec{a}\cdot\vec{b})\vec{a}$.

We could have obtain this formula with standard trigonometry:



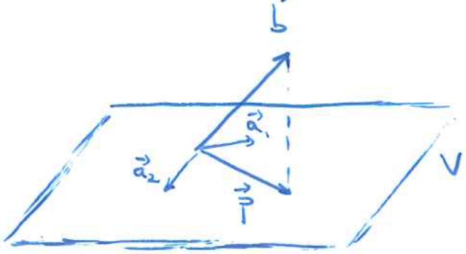
$$\left.\begin{array}{l} \cos\theta = \dfrac{\|\vec{p}\|}{\|\vec{b}\|} \Rightarrow \|\vec{p}\| = \|\vec{b}\|\cos\theta \\ \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta \underset{\|\vec{a}\|=1}{\uparrow}{=} \|\vec{b}\|\cos\theta \end{array}\right\} \Rightarrow$$

$\Rightarrow \|\vec{p}\| = \vec{a}\cdot\vec{b}$.

So the projection vector is $\vec{p} = \|\vec{p}\|\vec{a} = (\vec{a}\cdot\vec{b})\vec{a}$.

again, because $\|\vec{a}\|=1$.

- Projection of a vector $\vec{b}$ onto a subspace V:



1) Choose a basis of V: e.g. $\{\vec{a}_1, \vec{a}_2\}$ for plane.

2) Create $A = \left[\vec{a}_1 \middle| \cdots \middle| \vec{a}_n\right]$

3) $\vec{p} = P\vec{b}$ with $P = A(A^TA)^{-1}A^T$.

<u>Remark</u>: We choose the basis $\{\vec{a}_1, \dots, \vec{a}_n\}$. And note that

$$A^TA = \begin{bmatrix} \vec{a}_1^T \\ \hline \vdots \\ \hline \vec{a}_n^T \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T\vec{a}_1 & \vec{a}_1^T\vec{a}_2 & \cdots & \vec{a}_1^T\vec{a}_n \\ \vec{a}_2^T\vec{a}_2 & \vec{a}_2^T\vec{a}_2 & \cdots & \vec{a}_2^T\vec{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_n^T\vec{a}_1 & \vec{a}_n^T\vec{a}_2 & \cdots & \vec{a}_n^T\vec{a}_n \end{bmatrix}$$

In the case of a plane,

$$A^TA = \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T\vec{a}_1 & \vec{a}_1^T\vec{a}_2 \\ \vec{a}_2^T\vec{a}_1 & \vec{a}_2^T\vec{a}_2 \end{bmatrix}.$$

So if we choose $\vec{a}_1$ and $\vec{a}_2$ with size one and perpendicular to each other, that is, $\vec{a}_1 \cdot \vec{a}_2 = 0$, $\|\vec{a}_1\| = \|\vec{a}_2\| = 1$, then

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

• <u>Def</u>: The vectors $\vec{q}_1, \dots, \vec{q}_n$ are orthonormal if

$$\vec{q}_i^{\,T} \vec{q}_j = \begin{cases} 0 & i \neq j \text{ (orthogonal)} \\ 1 & i = j \text{ (unit vectors, i.e., } \|\vec{q}_i\| = 1) . \end{cases}$$

• <u>Remark</u>: $\boxed{\begin{matrix} Q \text{ is a matrix with} \\ \text{orthonormal columns} \end{matrix} \iff Q^T Q = I}$

$\hookrightarrow Q = \left[\vec{q}_1 \,\middle|\, \cdots \,\middle|\, \vec{q}_n \right]$

$$Q^T Q = \begin{bmatrix} \vec{q}_1^{\,T} \vec{q}_1 & \vec{q}_1^{\,T} \vec{q}_2 & \cdots & \vec{q}_1^{\,T} \vec{q}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{q}_n^{\,T} \vec{q}_1 & \vec{q}_n^{\,T} \vec{q}_2 & \cdots & \vec{q}_n^{\,T} \vec{q}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

• <u>Remark</u>: If Q is in addition square, then $Q^{-1} = Q^T$

$\hookrightarrow$ In this case, the matrix is called <u>orthogonal</u>.

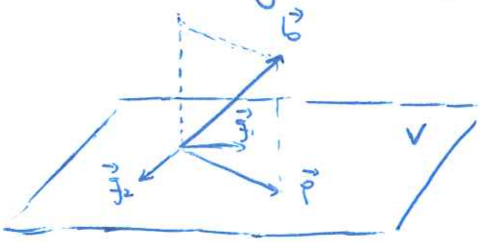
• <u>Remark</u>: For Q rectangular, QQ^T is not the identity:

Ex:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \to \quad Q^T Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (We knew it)}$$

$$QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

• Let's go back: projection using orthonormal basis:

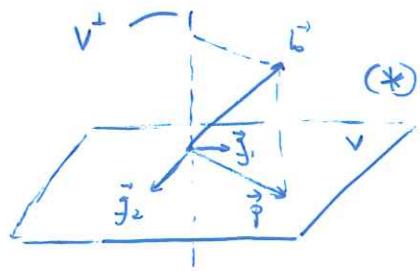


1) We choose an orthonormal basis of V: $\{\vec{q}_1, \dots, \vec{q}_n\}$

2) Create $Q = [\vec{q}_1 | \dots | \vec{q}_n]$.

3) Then, $\vec{p} = P\vec{b}$, with $P = Q(Q^TQ)^{-1}Q^T = \underbrace{QQ^T}$

(This is not the identity in general)



• Particular case of the plane:

$$\vec{p} = \alpha_1 \vec{q}_1 + \alpha_2 \vec{q}_2 \quad , \quad \alpha_1, \alpha_2 ?$$

$$\left.\begin{array}{l} \to \vec{q}_1 \cdot \vec{p} = \alpha_1 \vec{q}_1 \cdot \vec{q}_1 + \alpha_2 \cancel{\vec{q}_1 \cdot \vec{q}_2} = \alpha_1 \Rightarrow \alpha_1 = \vec{q}_1 \cdot \vec{p} \\ \to \vec{q}_2 \cdot \vec{p} = \alpha_1 \cancel{\vec{q}_2 \cdot \vec{q}_1} + \alpha_2 \vec{q}_2 \cdot \vec{q}_2 = \alpha_2 \Rightarrow \alpha_2 = \vec{q}_2 \cdot \vec{p} \end{array}\right\} \Rightarrow$$

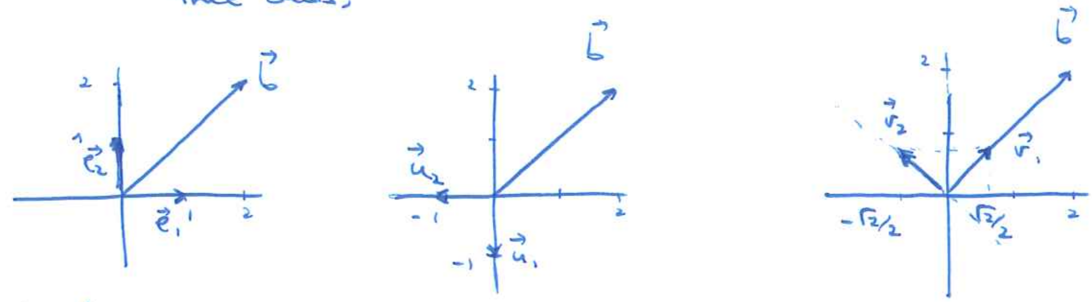
$$\Rightarrow \vec{p} = (\vec{q}_1 \cdot \vec{p})\,\vec{q}_1 + (\vec{q}_2 \cdot \vec{p})\,\vec{q}_2$$

That is, in an orthonormal basis, the projection into a subspace is the sum of the projections in each direction.

($\rightsquigarrow$ Notice in (*) that $\vec{b}$ = projection of $\vec{b}$ onto V + projection of $\vec{b}$ onto $V^\perp$

↳ so $\vec{p} = \vec{b}$ − projection of $\vec{b}$ onto $V^\perp$)

Example: Consider the vector $\vec{b}$ corresponding to the point $(2,2)$, and three basis,



$E = \{\vec{e}_1, \vec{e}_2\}$ $\qquad$ $U = \{\vec{u}_1, \vec{u}_2\}$ $\qquad$ $V = \{\vec{v}_1, \vec{v}_2\}$

E, U, V are orthonormal basis.

It is easy to see that

$\vec{b} = 2\vec{e}_1 + 2\vec{e}_2 \rightarrow \vec{b}|_E = (2,2)$ (coordinates of $\vec{b}$ in the basis E).

$\vec{b} = -2\vec{u}_1 - 2\vec{u}_2 \rightarrow \vec{b}|_U = (-2,-2)$ (" " " " " " U)

$\vec{b} = \sqrt{8}\,\vec{v}_1 \quad \rightarrow \quad \vec{b}|_V = (\sqrt{8}, 0)$ (" ... V)

Note that:

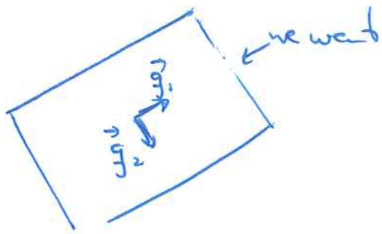
$\vec{b} = (\vec{b}\cdot\vec{e}_1)\vec{e}_1 + (\vec{b}\cdot\vec{e}_2)\vec{e}_2$, $\quad \vec{b} = (\vec{b}\cdot\vec{u}_1)\vec{u}_1 + (\vec{b}\cdot\vec{u}_2)\vec{u}_2$,

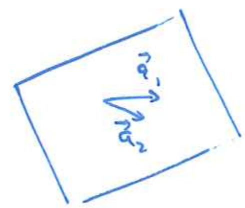
$\vec{b} = (\vec{b}\cdot\vec{v}_1)\vec{v}_1 + (\vec{b}\cdot\vec{v}_2)\vec{v}_2$

This is true in <u>all</u> <u>orthonormal</u> basis: a vector is the sum of its projections onto the basis vectors.

- In summary, orthonormal basis are great, but how to obtain them?

$\underline{\text{Ex}}$: Let $x+y+z=0$ plane in $\mathbb{R}^3$. Find an orthonormal basis.





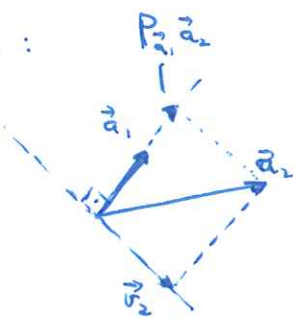
What we know: $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \rightsquigarrow \left\{ \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{a}_1}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{a}_2} \right\}$

But $\vec{a}_1 \cdot \vec{a}_2 \neq 0$

From $\{\vec{a}_1, \vec{a}_2\}$, we will obtain a new basis $\{\vec{q}_1, \vec{q}_2\}$ with the same span.

Gram-Schmidt Algorithm

Idea:



1) Take $\vec{v}_1 = \vec{a}_1$
2) Project $\vec{a}_2$ onto $\vec{a}_1 \to$ we obtain the component of $\vec{a}_2$ parallel to $\vec{a}_1$:
$$P_{\vec{a}_1}\vec{a}_2 = \frac{\vec{a}_1^T \vec{a}_2}{\vec{a}_1^T \vec{a}_1} \vec{a}_1$$
3) Then $\vec{v}_2 = \vec{a}_2 - P_{\vec{a}_1}\vec{a}_2 = \vec{a}_2 - \frac{\vec{a}_1^T \vec{a}_2}{\vec{a}_1^T \vec{a}_1} \vec{a}_1$ is orthogonal to $\vec{v}_1$.

4) Divide by their length to make them orthonormal:

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \quad \vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$$

• <u>General</u>: Given basis $\{\vec{a}_1, \dots, \vec{a}_k\}$,

1) Set $\vec{v}_1 = \vec{a}_1$

2) Set $\vec{v}_2 = \vec{a}_2 - \left(\frac{\vec{v}_1^T \vec{a}_2}{\vec{v}_1^T \vec{v}_1}\right)\vec{v}_1$

3) Set $\vec{v}_3 = \vec{a}_3 - \left(\frac{\vec{v}_1^T \vec{a}_3}{\vec{v}_1^T \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{v}_2^T \vec{a}_3}{\vec{v}_2^T \vec{v}_2}\right)\vec{v}_2$

⋮

k) Set $\vec{v}_k = \vec{a}_k - \left(\frac{\vec{v}_1^T \vec{a}_k}{\vec{v}_1^T \vec{v}_1}\right)\vec{v}_1 - \dots - \left(\frac{\vec{v}_{k-1}^T \vec{a}_k}{\vec{v}_{k-1}^T \vec{v}_{k-1}}\right)\vec{v}_{k-1}$

To conclude, set $\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \dots, \vec{q}_k = \frac{\vec{v}_k}{\|\vec{v}_k\|}$

→ (Continuation of example)

$\vec{a}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{a}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow$

1) $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

2) $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\|\vec{v}_1\|^2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} =$

$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ (Check that $\vec{v}_1 \cdot \vec{v}_2 = 0$!).

And $\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$

$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$

$\boxed{\begin{array}{c}\text{MATH 312}\\ \text{LECTURE 12}\end{array}}$: Linear Transformations (~ Sec. 8.1, 8.2)

I.) Introduction

Def. Let $\mathcal{U}$, V be vector spaces. A function T from $\mathcal{U}$ to V,

$$\left.\begin{array}{l} T: \mathcal{U} \longrightarrow V \\ \vec{u} \longmapsto \vec{v} = T(\vec{u}) \end{array}\right\} \text{ is called a linear transformation if}$$

1) $T(\vec{u}_1 + \vec{u}_2) = T(\vec{u}_1) + T(\vec{u}_2)$

2) $T(\lambda \vec{u}) = \lambda T(\vec{u})$

for all $\vec{u}_1, \vec{u}_2 \in \mathcal{U}$, λ scalar.

Remarks:

1) $T(\vec{0}) = \vec{0}$

2) $T(\alpha \vec{u}_1 + \beta \vec{u}_2) = \alpha T(\vec{u}_1) + \beta T(\vec{u}_2)$.

Example: Which of these transformations are linear? $a, b \in \mathbb{R}$

a) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ a \end{bmatrix}$ ✓ b) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ab$ ×

c) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ × d) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a - b$ ✓

• Notice that in a) and d) T is given by a matrix:

a) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$, d) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = a - b$

Example: Which are linear?

1) $T : \mathbb{R}^3 \to \mathbb{R}^3$
$\vec{u} \mapsto T(\vec{u}) = u_1 + u_2 + u_3$

2) $T : \mathbb{R}^3 \to \mathbb{R}$
$\vec{u} \mapsto \|\vec{u}\|$

Example: Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Assume we know that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = -1 \quad \text{and} \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = 2,$$

what is $T\left(\begin{bmatrix}-2\\3\\0\end{bmatrix}\right)$? (Ans: $= -2(-1) + 3(2) = 8$.

Example: The derivative

Let $\mathcal{U}$ the vector space of polynomials with degree no more than 2,
V " " " " " " " " " " " 1.

That is, $\mathcal{U} = \{p \text{ polyn.} : p(x) = a + bx + cx^2,\ a, b, c \in \mathbb{R}\}$

$V = \{p \text{ polyn.} : p(x) = a + bx,\ a, b \in \mathbb{R}\}$

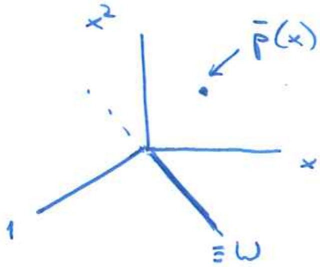
We can choose basis for $\mathcal{U}$ and V:

$B_{\mathcal{U}} = \{1, x, x^2\}$

$B_V = \{1, x\}$

• So now we can "draw" polynomials!

$$p(x) = a + bx + cx^2 \sim (a, b, c)$$



$$\bar{p}(x) = x + x^2 \rightsquigarrow (0, 1, 1)$$

V is the plane "1, x" (subspace of $\mathcal{U}$).

$$W = \{ p \text{ pol} : p(x) = a + ax \}$$

Let's consider the derivative of polynomials in $\mathcal{U}$:

$$\left.\begin{aligned} T : \mathcal{U} &\to \mathcal{U} \\ p(x) &\mapsto T(p(x)) = \frac{d}{dx} p(x) \end{aligned}\right\}$$

$$T\left(a + bx + cx^2\right) = b + 2cx$$

We can write this using matrices:

$$\left.\begin{aligned} p(x) = a + bx + cx^2 &\to p\big|_{B_u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ T(p(x)) = b + 2cx &\to T(p(x))\big|_{B_u} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix} \end{aligned}\right\} \text{ that is}$$

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \cancel{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$$

Let's check if: $p(x) = 4x^2 + 3$

$$\frac{d}{dx} p(x) = 8x \checkmark \; ; \; T\left(\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} \to 8x //.$$

<u>Remark</u>: The matrix that represents the transformation is different if we choose different basis for U

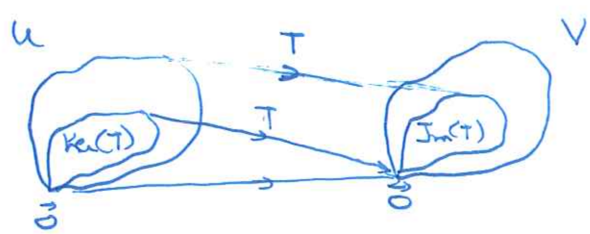
<u>Def</u>: Kernel of a linear transformation.

Let $\left.\begin{matrix} T: U \to V \\ \vec{u} \mapsto \vec{v} = T(\vec{u}) \end{matrix}\right\}$ lin. transf. Its kernel is defined as

$$\ker(T) = \{ \vec{u} \in U : T(\vec{u}) = \vec{0} \}$$

<u>Def</u>: Range of a lin. transf. (or Image).

$$\operatorname{Im}(T) = \{ \vec{v} \in V : \exists \vec{u} \in U \text{ with } T(\vec{u}) = \vec{v} \}.$$



→ So if the linear transformation is given by a matrix, $T(\vec{x}) = A\vec{x}$,

then $\ker(T) \equiv N(A)$

$\operatorname{Im}(T) \equiv C(A)$.

• For the derivative, what is the Kernel?

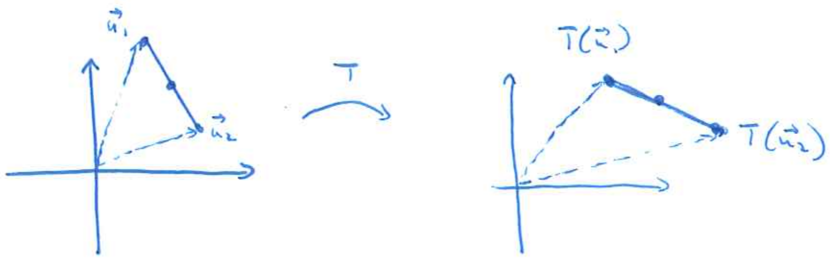
$\mathrm{Ker}(T)$ = polynomials of degree zero (constants).

$$\hookrightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{basis of}]{} N(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

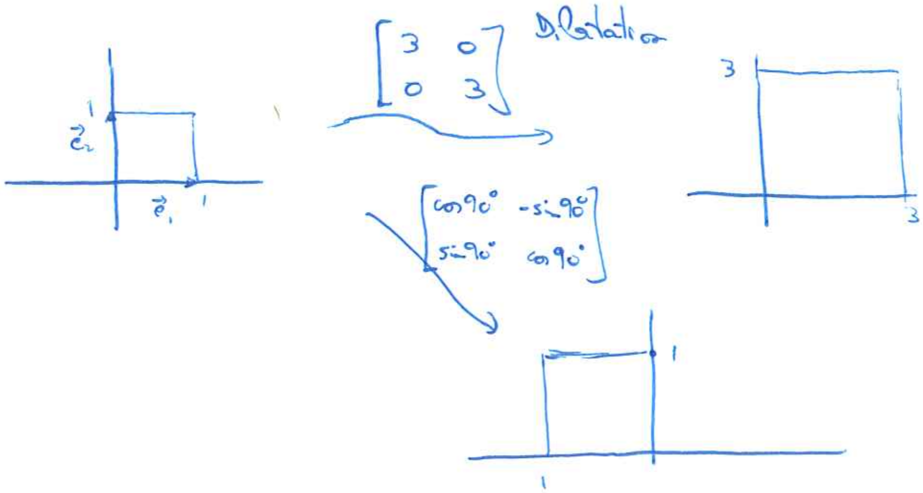
$\mathrm{Im}(T)$ = polynomials of degree one or less $\rightarrow V$

$$\hookrightarrow C(A) \rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

<u>Example</u>: Image transformation.



The middle points goes to the middle point: $T\left(\frac{1}{2}(\vec{u}_1+\vec{u}_2)\right) = \frac{1}{2}\left(T(\vec{u}_1)+T(\vec{u}_2)\right)$.



You only need to see where $\vec{e}_1$ and $\vec{e}_2$ goes!

II] The matrix of a linear transformation.

Let $\left.\begin{array}{l} T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ \quad \vec{x} \longmapsto T(\vec{x}) = A\vec{x} \end{array}\right\}$ with A $m \times n$ matrix.

For example.

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ 5 & 0 & 1 & 0 \end{bmatrix}, \text{ so } T_A : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

Choose the canonical basis for $\mathbb{R}^4$ and $\mathbb{R}^3$ (i.e. $\mathbb{R}^4 \rightsquigarrow \{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{e}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{e}_2}, \dots \}$).

What is $T_A(\vec{e}_1)$?

$$T_A(\vec{e}_1) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ 5 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$\llcorner$ 1st column of A.

So,

$$A = \left[T(\vec{e}_1) \middle| \dots \middle| T(\vec{e}_4) \right]$$

Example. $a + bx + cx^2 \rightsquigarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in basis $\{1, x, x^2\}$, $T \equiv$ derivation.

$$\left.\begin{array}{lll} 1 \xrightarrow{T} 0 & \rightsquigarrow & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ x \longmapsto 1 & \rightsquigarrow & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ x^2 \longmapsto 2x & \rightsquigarrow & \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \end{array}\right\} \rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

• Change of basis:

Let $T: \mathcal{U} \to \mathcal{U}$, $\vec{u} \mapsto T(\vec{u}) = \vec{u}$ } but choose a different basis of $\mathcal{U}$ for the input and output spaces.

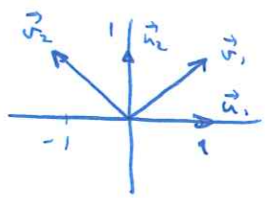
Input basis $B_i = \{\vec{u}_1, \dots, \vec{u}_n\}$
Output basis $B_o = \{\vec{v}_1, \dots, \vec{v}_n\}$ (same number of vectors!).

Assume we know

$$\left.\begin{aligned} \vec{u}_1 &= m_{11}\vec{v}_1 + \dots + m_{1n}\vec{v}_n \\ \vec{u}_2 &= m_{21}\vec{v}_1 + \dots + m_{2n}\vec{v}_n \\ &\vdots \\ \vec{u}_n &= m_{n1}\vec{v}_1 + \dots + m_{nn}\vec{v}_n \end{aligned}\right\} \to \text{Define } M = \begin{bmatrix} m_{11} & m_{21} & \dots & m_{n1} \\ m_{12} & & & \vdots \\ \vdots & & \ddots & \vdots \\ m_{1n} & \dots & & m_{nn} \end{bmatrix} \text{ change of basis matrix}$$

↑ coordinates of $\vec{u}_1$ in B_o ↑ coordinates of $\vec{u}_n$ in B_o

Example:



Given $\vec{b} = 3\vec{u}_1 + 2\vec{u}_2$ (i.e., $\vec{b}|_u = (3, 2)$), what are the coordinates of $\vec{b}$ in V?

$$\left.\begin{aligned} \vec{v}_1 &= \vec{u}_1 + \vec{u}_2 \\ \vec{v}_2 &= -\vec{u}_1 + \vec{u}_2 \end{aligned}\right\} \to \left.\begin{aligned} \vec{u}_1 &= \tfrac{1}{2}\vec{v}_1 - \tfrac{1}{2}\vec{v}_2 \\ \vec{u}_2 &= \tfrac{1}{2}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2 \end{aligned}\right\} \Rightarrow M = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$\text{So } \vec{b}|_V = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix} \to \vec{b} = \frac{5}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$$

↳ Remark: $\vec{v}_1|_V = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of course!

Example: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \{1, x, x^2\}$ → U basis

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \{1, x-1, (x-1)^2\}$ → V basis

Given $p(x) = 2 + x + 2x^2$, how to write it in the V basis?

$$\left.\begin{aligned} \vec{u}_1 &= \vec{v}_1 \\ \vec{u}_2 &= \vec{v}_1 + \vec{v}_2 \\ \vec{u}_3 &= \vec{v}_1 + 2\vec{v}_2 + \vec{v}_3 \end{aligned}\right\} \to M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$(x-1)^2 =$
$= x^2 - 2x + 1$

And therefore $p(x)\Big|_V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$

that is, $2 + x + 2x^2 = 5 + 5(x-1) + 2(x-1)^2$ Taylor expansion at $x=1$!!

Exercise: $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ $\Big|$ T linear.

$T(\vec{u}_1) = \vec{v}_1$, $T(\vec{u}_2) = T(\vec{u}_3) = \vec{v}_1 + \vec{v}_3$

$\begin{cases} \text{Find the matrix } A \text{ such that } T(\vec{u}) = A\vec{u} \\ \text{Find } T(\vec{u}_1 + \vec{u}_2 + \vec{u}_3). \end{cases}$

Sol:

We know that $A = \left[T(\vec{u}_1) \,\middle|\, T(\vec{u}_2) \,\middle|\, T(\vec{u}_3) \right]$ with $T(\vec{u}_1), T(\vec{u}_2), T(\vec{u}_3)$

written in the output basis V.

So,

$$T(\vec{u}_1)\big|_V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad T(\vec{u}_2)\big|_V = T(\vec{u}_3)\big|_V = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Therefore, since

$$(\vec{u}_1 + \vec{u}_2 + \vec{u}_3)\big|_{\mathcal{U}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow T(\vec{u}_1 + \vec{u}_2 + \vec{u}_3)\big|_V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \begin{matrix} \leftarrow \\ \\ \leftarrow \end{matrix}$$

• We could have done this without the matrix:

$$T(\vec{u}_1 + \vec{u}_2 + \vec{u}_3) = T(\vec{u}_1) + T(\vec{u}_2) + T(\vec{u}_3) = \vec{v}_1 + (\vec{v}_1 + \vec{v}_3) + (\vec{v}_1 + \vec{v}_3) =$$

$$= \underline{\underline{3}}\vec{v}_1 + \underline{\underline{2}}\vec{v}_3.$$