Quizz (10) (LECTURE 15)

- Given a basis, one can always find an orthonormal one with the same span → True: Gram-Schmidt

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{u} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

- For any basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of the vector space U, it holds that $\mathbf{u} = (\mathbf{u}_1 \cdot \mathbf{u})\mathbf{u}_1 + (\mathbf{u}_2 \cdot \mathbf{u})\mathbf{u}_2 + (\mathbf{u}_3 \cdot \mathbf{u})\mathbf{u}_3$, $(\mathbf{u} \in U)$ False: (True for orthonormal basis: $\vec{u} = \alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 + \alpha_3\vec{u}_3$ ↳ $\vec{u}\cdot\vec{u}_1 = \alpha_1, \ldots$)

- If an m by n matrix Q has orthonormal colums, then $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$ True: $\|Q\vec{x}\| = (Q\vec{x}\cdot Q\vec{x})^{1/2} = (\vec{x}^T Q^T Q\vec{x})^{1/2} = (\vec{x}^T\vec{x})^{1/2} = \|\vec{x}\|$.

4) - If $det(A) = -1$ for some square matrix A, then there is some $\mathbf{b}$ such that $Ax = \mathbf{b}$ has infinitely many solutions False. $det(A) \neq 0 \Rightarrow A^{-1}$ exists $\Rightarrow A\vec{x} = \vec{b}$ has a unique sol.

3) - If $det(A^2) = 1$ for some square matrix A, then A has all eigenvalues equal to 1 or -1 False ↳ $A = \begin{bmatrix}2 & 0\\0 & 1/2\end{bmatrix}$

5) - If $A^2 = 0$ for some square matrix A, then all the eigenvalues of A are zero. True: $A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}$ ↳ $A^2\vec{x} = \vec{0} \Rightarrow \lambda^2 = 0 \Rightarrow \underline{\underline{\lambda = 0}}$

- A 3 by 3 matrix with eigenvalues $0, 0, 1$ always verifies that $rank(A) = 1$ False ↳ $\lambda = 0$ might not give two eigenvectors: $\begin{bmatrix}0 & 0 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$

- A 3 by 3 matrix with eigenvalues 0,1,-1 is always diagonalizable True: distinct eigenvalues give l.i. eigenvectors.

- An orthogonal matrix (i.e., orthonormal columns and rows) always has eigenvalues with modulo 1: $|\lambda| = 1$ True: $Q\vec{x} = \lambda\vec{x}$

↳ $\|Q\vec{x}\| = \|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$, $\|Q\vec{x}\| = \|\vec{x}\|$ $\Rightarrow |\lambda| = 1$

- Two eigenvectors corresponding to two different eigenvalues are always linearly independent.

True.

D.Ad.: $c_1\vec{u}_1 + c_2\vec{u}_2 = 0 \rightarrow c_1A\vec{u}_1 + c_2A\vec{u}_2 = 0$

$c_1\lambda_1\vec{u}_1 + c_2\lambda_2\vec{u}_2 = 0$

$c_1\lambda_2\vec{u}_1 + c_2\lambda_2\vec{u}_2 = 0$; $\rightarrow c_1(\lambda_1 - \lambda_2)\vec{u}_1 = 0 \Rightarrow c_1 = 0$.

$c_1\lambda_1\vec{u}_1 + c_2\lambda_1\vec{u}_2 = 0 \longrightarrow c_2(\lambda_1 - \lambda_2)\vec{u}_2 = 0 \Rightarrow c_2 = 0$

$\rightarrow \cancel{\ }$.

MATH 312
LECTURE 15

: Linear differential equations (with constant coefficients)

IV] Linear differential equations with constant coeff.

Consider the 1-d differential equation

$$\left.\begin{aligned} x'(t) &= \alpha x(t) \\ x(0) &= x_0 \end{aligned}\right\} \qquad \left(x'(t) \equiv \frac{dx}{dt}(t) \right)$$

$x(0) = x_0 \longrightarrow$ (Initial condition)

To solve it, we simply integrate it:

$$\int_0^t \frac{x'(\tau)}{x(\tau)}\, d\tau = \int_0^t \alpha\, d\tau \Rightarrow \ln x(\tau)\Big]_0^t = \alpha t \Rightarrow$$

$$\Rightarrow \ln(x(t)) - \ln(x(0)) = \ln\left(\frac{x(t)}{x_0}\right) = \alpha t \Rightarrow \boxed{x(t) = x_0 e^{\alpha t}}$$

The <u>sign</u> of the constant α tell us if the solution grows exponentially to infinity or decreases exponetially to zero.

- <u>Remark</u>: We can always check the solution of a diff. eq.:

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left(x_0 e^{\alpha t}\right) = x_0 \alpha e^{\alpha t} = \alpha x(t) \;\; \text{(satisfies the eq.)}$$

· The previous equation provides a simple (but silly) model for growth population:

$$\underbrace{p'(t)}_{\text{rate of growth at time } t} = \underbrace{\alpha}_{\text{fraction of reproducing people}}\ \underbrace{p(t)}_{\text{current population}}$$

"The more we are, the more kids we have". ($\alpha \in [0,1]$)

But this always gives $p(t) = p_0 e^{\alpha t}$! (Exponential growth).

· We can improve the model taking into account the resourses:

$r(t) \equiv$ resourses at time t
$p(t) \equiv$ population at time t

and let's assume that

$p'(t)$ is proportional to $r(t)$ and $p(t)$ (increasing)
$r'(t)$ " " to $p(t)$ (decreasing)

then,

$$\left.\begin{aligned} p'(t) &= \alpha p(t) + \beta r(t) \\ r'(t) &= -\gamma\, p(t) \end{aligned}\right\} \quad \text{(with } \alpha, \beta, \gamma > 0 \text{ constants).}$$

We can write this in matrix form:

$$\begin{bmatrix} p(t) \\ r(t) \end{bmatrix}' = \begin{bmatrix} \alpha & \beta \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ r(t) \end{bmatrix}$$

- **Def**: We call linear system of differential equations with constant coefficients those of the form

  $\vec{x}'(t) = A\vec{x}(t)$ with A $n \times n$ matrix and

  $\vec{x}(0) = \vec{x}_0$ initial condition.

How to solve it?

First, note that if $A = D$ diagonal, then the system $\vec{x}'(t) = D\vec{x}(t)$ is in reality n separated equations:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \sim \begin{array}{l} x_1'(t) = \lambda_1 x_1(t) \Rightarrow x_1(t) = x_1(0)e^{\lambda_1 t} \\ x_2'(t) = \lambda_2 x_2(t) \Rightarrow x_2(t) = x_2(0)e^{\lambda_2 t} \end{array}$$

or in matrix notation,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}.$$

This motivates the following definition:

• Def: Exponential matrix e^A

$$e^A \overset{\text{def.}}{=} I + A + \frac{A^2}{2!} + \dots = \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

(recall $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ when $x \in \mathbb{R}$)

Remarks:

1) If $A = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $e^D = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$

$$\hookrightarrow e^D = \sum_{j=0}^{\infty} \frac{1}{j!} D^j = \sum_{j=0}^{\infty} \begin{bmatrix} \frac{\lambda_1^j}{j!} & 0 \\ 0 & \frac{\lambda_2^j}{j!} \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{\infty} \frac{\lambda_1^j}{j!} & 0 \\ 0 & \sum_{j=0}^{\infty} \frac{\lambda_2^j}{j!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$$

2) If A is diagonalizable, i.e, $A = SDS^{-1}$, then

$$\left\| e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!} = \sum_{j=0}^{\infty} \frac{1}{j!} SD^jS^{-1} = S\left(\sum_{j=0}^{\infty} \frac{D^j}{j!}\right) S^{-1} = Se^DS^{-1} \right\|$$

3) $\frac{d}{dt}(e^{At}) = Ae^{At}$

$\hookrightarrow \frac{d}{dt}\left(\sum_{j=0}^{\infty} \frac{A^j}{j!} t^j\right) = \frac{d}{dt}\left(I + \sum_{j=1}^{\infty} \frac{A^j t^j}{j!}\right) = \sum_{j=1}^{\infty} \frac{A^j j}{j!} t^{j-1} =$

$= A \sum_{j=1}^{\infty} \frac{A^{j-1}}{(j-1)!} t^{j-1} = A \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k = Ae^{tA}$ //.

Conclusion: The system $\left.\begin{array}{l}\vec{x}'(t) = A\vec{x}(t)\\ \vec{x}(0) = \vec{x}_0\end{array}\right\}$ is solved by

$$\vec{x}(t) = e^{At}\vec{x}_0$$

Proof: Let's check that $\vec{x}(t) = e^{At}\vec{x}_0$ verifies the equation:

$$\vec{x}'(t) = (e^{At}\vec{x}_0)' = Ae^{At}\vec{x}_0 = A\vec{x}(t) //$$

(and obviously verifies the initial condition $\vec{x}(0) = e^{\overset{\text{zero matrix}}{0}}\vec{x}_0 = I\vec{x}_0 = \vec{x}_0$)

- Remark: The solution $\vec{x}(t) = e^{At}\vec{x}_0$ is valid for any A //.

Example: Solve
$$\left.\begin{aligned} x_1'(t) &= 7x_1(t) + 5x_2(t) \\ x_2'(t) &= -10x_1(t) - 8x_2(t) \end{aligned}\right\}$$

with $x_1(0) = 1$, $x_2(0) = 2$.

Solution: $\vec{x}\,'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}}_{A} \vec{x}(t)$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\vec{x}(t) = e^{At}\vec{x}(0) \rightarrow$ Find e^{At}.

We first try to diagonalise A:

$$A - \lambda I = \begin{bmatrix} 7-\lambda & 5 \\ -10 & -8-\lambda \end{bmatrix} \rightarrow -(7-\lambda)(8+\lambda) + 50 = \lambda^2 + \lambda - 6 = 0 \Leftrightarrow$$

$$\Leftrightarrow \lambda = \frac{-1 \pm \sqrt{25}}{2} \begin{array}{l} \nearrow \lambda_1 = -3 \\ \searrow \lambda_2 = 2 \end{array} \quad (\Rightarrow \text{we can diagonalise})$$

Eigenvectors:

$$A - \lambda_1 I = \begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, $S = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$, $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

and so

$$e^{At} = S e^{Dt} S^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

<u>Remark</u>: The solution can be written as follows

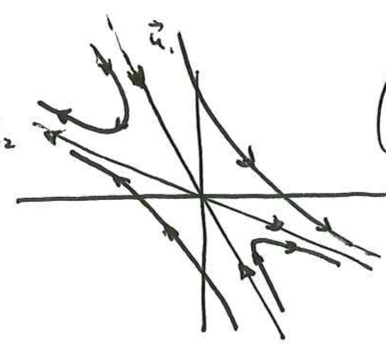
$$\vec{x}(t) = e^{At}\vec{x}_0 = \underset{S}{\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}} \underset{D}{\begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{bmatrix}} \underbrace{\underset{S^{-1}}{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}} \underset{\vec{x}_0|_E}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}}_{\vec{x}_0|_S} =$$

$$= \begin{bmatrix} -e^{-3t} & e^{2t} \\ 2e^{-3t} & -e^{2t} \end{bmatrix} \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\vec{x}_0|_S} = 3e^{-3t} \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\vec{u}_1} + 4e^{2t} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{u}_2}$$

coordinates of $\vec{x}_0$ in $\{\vec{u}_1, \vec{u}_2\}$

This shows us how the solution evolves in time: the $\vec{u}_1$ component goes to zero since its corresponding eigenvector is negative.

Phase portrait:



(Saddle point)
↳ Always the case when $\lambda_1 \cdot \lambda_2 < 0$
(i.e., distinct signs)

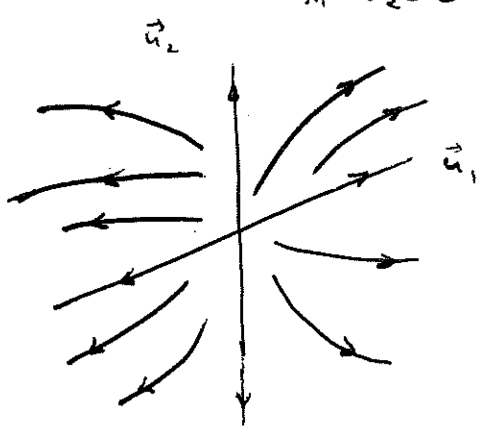
<u>Remark</u>: If A is diagonalizable, the solution can always be written as

$$\vec{x}(t) = Se^{Dt}S^{-1}\vec{x}_0 = \left[\vec{u}_1 \mid \vec{u}_2\right]\begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{bmatrix} \vec{x}_0|_S =$$

$$= c_1 e^{\lambda_1 t}\vec{u}_1 + c_2 e^{\lambda_2 t}\vec{u}_2 \quad , \text{ where } \vec{x}_0|_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

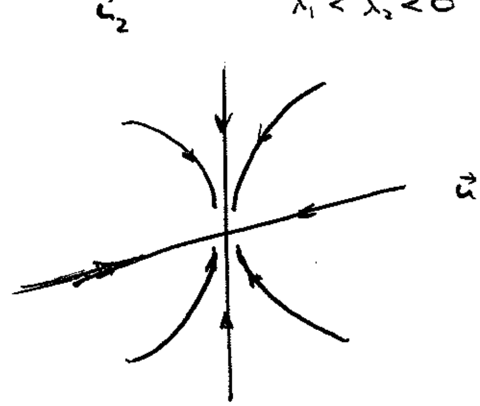
Depending on the <u>signs</u> of λ_1, λ_2, different phase portraits arise

$\lambda_1 > \lambda_2 > 0$



(everything grows, but faster in $\vec{u}_1$ direction)

$\lambda_1 < \lambda_2 < 0$



(everything goes to zero, but the $\vec{u}_1$ component dissapears faster)

⋮

MATH 312 LECTURE 16	: The Spectral Theorem

From last lecture, we know that all linear systems of diff. eqs. with constant coeff.,

$$\left.\begin{aligned} \vec{x}'(t) &= A\vec{x}(t) \\ \vec{x}(0) &= \vec{x}_0 \end{aligned}\right\}$$

are solved by $\vec{x}(t) = e^{At}\vec{x}_0$, where $e^{At} = \sum_{j=0}^{\infty} \frac{A^j t^j}{j!}$.

→ Remark: When $A = SDS^{-1}$ (diagonalizable), then $e^{At} = Se^{Dt}S^{-1}$ and thus

$$\vec{x}(t) = \left[\vec{u}_1 \,|\, \vec{u}_2\right]\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{bmatrix}(S^{-1}\vec{x}_0) =$$

$$= \left[e^{\lambda_1 t}\vec{u}_1 \,\middle|\, e^{\lambda_2 t}\vec{u}_2\right]\underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{coordinates of } \vec{x}_0 \text{ in } S.} = c_1 e^{\lambda_1 t}\vec{u}_1 + c_2 e^{\lambda_2 t}\vec{u}_2$$

Remark: In an ODE course, many times you assume this as a solution (by "faith"), and then find c_1, c_2 by imposing the initial condition:

$$\vec{x}(0) = c_1 e^{\lambda_1 0}\vec{u}_1 + e^{\lambda_2 0}\vec{u}_2 \Rightarrow \vec{x}_0 = c_1\vec{u}_1 + c_2\vec{u}_2 \overset{\text{solve}}{\leadsto} c_1, c_2 \,/\!/.$$

• Exercise: 2^{nd} order linear differential equation.

Solve $x''(t) = -x(t) + 2x'(t)$ with $x(0) = 2$, $x'(0) = 1$.

Sol.: Right now, we only know how to solve first order linear systems.

Let's convert this 2^{nd} order diff. eq. into a 1^{st} order system:

Denote
$$\left.\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x'(t) \end{aligned}\right\}$$

Then, $x_1'(t) = x'(t) = x_2(t)$

$$x_2'(t) = \underbrace{x''(t) = -x(t) + 2x'(t)}_{\text{use the equation}} = \underbrace{-x_1(t) + 2x_2(t)}_{\text{introduce the new variables}}$$

So, in matrix form,

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}'}_{\vec{x}'(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\vec{x}(t)}$$
and thus $\vec{x}(t) = e^{At}\vec{x}_0$.

Remark: This idea can be applied to any n-order lin. diff. equation with constant coeff. to produce ~~an~~ ~~n~~ a 1^{st} order lin. system (of n equations)

$$a_0 x(t) + a_1 x'(t) + \dots + a_n x^{n)}(t) = 0$$

↳ Define
$$\left.\begin{aligned} x_1(t) &= x(t) \\ x_2(t) &= x'(t) \\ &\vdots \\ x_n(t) &= x^{n-1)}(t) \end{aligned}\right\} \quad \vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \vdots & & \ddots & \vdots \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \dots & & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Let's finish the exercise. We need e^{At}.

Try to diagonalise:

Eigenvalues of A: $\det(A - \lambda I) = 0 \Leftrightarrow -\lambda(2-\lambda) + 1 = 0 \Leftrightarrow$

$$\Leftrightarrow \lambda^2 - 2\lambda + 1 = 0 \Leftrightarrow (\lambda - 1)^2 = 0$$

$$\lambda = 1 \text{ (repeated)}$$

Eigenvectors: $(A - I)\vec{u} = \vec{0} \rightarrow \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ has rank 1! Only one eigenvector.

So A cannot be diagonalised.

We have to use defintion of $e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$

However,

$$A^2 = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}, \dots \quad \text{infinite terms?!}$$

- <u>Nice fact</u>: (Cayley-Hamilton Theorem)

$$(A-I)^2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

[A matrix A always satisfies its own characteristic equation: $\det(A - \lambda I) = 0$]

(here $\det(A - \lambda I) = (\lambda - 1)^2 = 0$)

So we can write $A = I + (A - I)$ and so,

$$e^{At} = e^{It + (A-I)t} \overset{(*)}{=} e^{It} e^{(A-I)t} =$$

$$= e^t I \left(I + (A-I)t + \underbrace{\cancel{(A-I)^2}}_{0} \frac{t^2}{2!} + \cancel{\dots} \right) =$$

$$= e^t \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} t \right) = e^t \begin{bmatrix} 1-t & t \\ -t & 1-t \end{bmatrix}$$

((*) $e^{A+B} = e^A e^B \iff A$ and B commute (i.e., $AB = BA$))

We conclude that

$$\vec{x}(t) = e^{At}\vec{x}_0 = e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 2-2t+t \\ -2t+1+t \end{bmatrix} = e^t \begin{bmatrix} 2-t \\ 1-t \end{bmatrix} =$$

$$= \underbrace{e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{x}_0} - t e^t \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{u}_1}$$

(of course $\vec{x}(0) = e^0 \vec{x}_0 - \vec{0} = \vec{x}_0$)

V) Difference equations (sequences)

$$\left.\begin{aligned} \vec{x}(k) &= A\vec{x}(k-1) \\ \vec{x}(0) &= \vec{x}_0 \end{aligned}\right\}$$

(... like differential equations with discrete time... bla blah...)

Solution: $\vec{x}(1) = A\vec{x}(0) = A\vec{x}_0$

$\vec{x}(2) = A\vec{x}(1) = AA\vec{x}_0 = A^2\vec{x}_0$

$\vdots$

$$\| \vec{x}(k) = A^k \vec{x}_0 \|$$

If $A = SDS^{-1}$, then

$$\vec{x}(k) = S D^k S^{-1}\vec{x}_0 = S \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} S^{-1} \vec{x}_0$$

<u>Example</u>: Virahanka (or Fibonacci) numbers.

Sequence 0, 1, 1, 2, 3, 5, 8, ..., N_k, N_{k+1}, N_{k+2}, ...

N_k ? $\frac{N_k}{N_{k-1}}$?

<u>Sol</u>: The sequence is defined by $N_{k+2} = N_{k+1} + N_k$.

This is a 2^{nd} order difference equation (as it involves two steps).

But we "already know" how to solve that problem: convert it into two 1^{st} order equations.

Denote

$$\vec{x}(k) = \begin{bmatrix} N_{k+1} \\ N_{k+2} \end{bmatrix}, \text{ then}$$

$$\underbrace{\vec{x}(k) = \begin{bmatrix} N_{k+1} \\ N_{k+2} \end{bmatrix}}_{\vec{x}(k)} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} N_k \\ N_{k+1} \end{bmatrix}}_{\vec{x}(k-1)} \quad \text{with } \vec{x}(0) = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So the solution is

$\vec{x}(k) = A^k \vec{x}_0 \longrightarrow A^k$? Diagonalize:

1) Eigenvalues: $\det(A-\lambda I)=0 \iff$

$$\iff \lambda^2-\lambda-1=0 \iff \lambda = \frac{1\pm\sqrt{5}}{2} \begin{cases} \lambda_1 = \frac{1+\sqrt{5}}{2} \\ \lambda_2 = \frac{1-\sqrt{5}}{2} \end{cases}$$

2) Eigenvectors: $(A-\lambda_1 I) = \begin{bmatrix} -\lambda_1 & 1 \\ 1 & 1-\lambda_1 \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ (note that $1+\lambda_1-\lambda_1^2=0$)

Analogously, $\vec{u}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$

In conclusion,

$$\vec{x}(k) = \underbrace{\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}}_{S} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} (S^{-1}\vec{x}_0) \quad \text{where } S^{-1}\vec{x}_0 = \vec{x}_0|_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

(just a matter of computing)

thus

$$\vec{x}(k) = \left[\lambda_1^k \vec{u}_1 \,\middle|\, \lambda_2^k \vec{u}_2 \right] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \lambda_1^k \vec{u}_1 + c_2 \lambda_2^k \vec{u}_2 .$$

Therefore,

$N_{k+1} = c_1 \lambda_1^{k+1} + c_2 \lambda_2^{k+1}$ (2nd component of $\vec{x}(k)$)

and

$$\frac{N_{k+2}}{N_{k+1}} = \frac{c_1\lambda_1^{k+1} + c_2\lambda_2^{k+1}}{c_1\lambda_1^k + c_2\lambda_2^k} = \frac{c_1\lambda_1 + c_2\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^k}{c_1 + c_2(\lambda_2/\lambda_1)^k} \Rightarrow \lim_{k\to\infty} \frac{N_{k+2}}{N_{k+1}} = \lambda_1 = \frac{1+\sqrt{5}}{2}$$

("golden ratio")

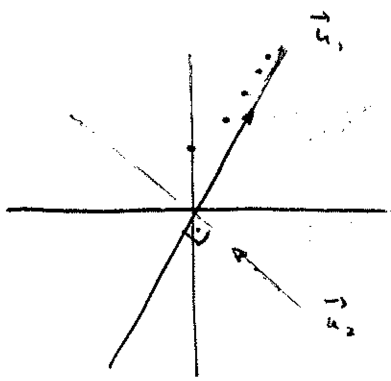
(because $\lambda_2 < \lambda_1$)

Forgetting about Virahanka or Fibonacci, we could think of

$$\left.\begin{array}{l}\vec{x}(k) = \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(k-1) \\ y(k-1) \end{bmatrix} \\ \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\end{array}\right\} \text{ as a system,}$$

whose solution $\vec{x}(k) = c_1 \lambda_1^k \vec{u}_1 + c_2 \lambda_2^k \vec{u}_2$ can be drawn as "time" evolves (k increases):

$\vec{x}(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x}(2) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, ...



The $\vec{u}_2$ component goes to zero as $|\lambda_2| < 1$.

→ Notice that $\vec{u}_1 \cdot \vec{u}_2 = 1 + \lambda_1 \lambda_2 = \frac{1+\sqrt{5}}{2} \frac{1-\sqrt{5}}{2} = 1 - 1 = 0$, that is, $\vec{u}_1 \perp \vec{u}_2$.

This is not a coincidence: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is symmetric with real entries.

VI/ The Spectral Theorem.

• **Theorem:** The Spectral Theorem
(or Principal Axis Theorem)

If M is a real symmetric $n \times n$ matrix, then it has real eigenvalues and the eigenvectors can be chosen to be orthonormal. That is,

M can always diagonalised as

$M = Q D Q^T$, where

$D \equiv$ diagonal with real eigenvalues.

$Q \equiv$ formed by n orthonormal eigenvectors.

Remarks:

1) Eigenvectors corresponding to eigenvalues go in the same order:

$$\begin{array}{l} \lambda_1 \leftrightarrow \vec{u}_1 \\ \lambda_2 \leftrightarrow \vec{u}_2 \\ \vdots \end{array} \rightsquigarrow Q = \left[\vec{u}_1 \mid \vec{u}_2 \mid \ldots\right], \quad D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$$

Note that if $M = QDQ^T$, then M is symmetric: $(QDQ^T)^T = QDQ^T$.

So $M = QDQ^T \iff M$ is symmetric

2) If the eigenvalues are distinct, then the eigenvectors are automatically perpendicular to each other.

↳ We just need to be careful to make them unit size. (i.e., orthonormal).

3) If an eigenvalue is repeated (let's say twice), then the nullspace of $A - \lambda I$ will have dimension 2 (or more if repeated more times...).

In this case we need to obtain an orthonormal basis of that nullspace (the "eigenspace").

↳ obtain a basis (as usual) and use Gram-Schmidt!

4) Q has orthonormal columns and it is square. Thus, it automatically has ~~orthogonal~~ orthonormal rows too. This is called orthogonal matrix. [(*) proof in -163-]

Q orthogonal $\Leftrightarrow Q^TQ = QQ^T = I$ (i.e., $Q^T = Q^{-1}$).

Q orthonormal columns $\Leftrightarrow Q^TQ = I$

Q orthonormal rows $\Leftrightarrow QQ^T = I$.

5) $M = QDQ^T = \left[\vec{q}_1 | \cdots | \vec{q}_n\right] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix} =$

$$= \left[\lambda_1 \vec{q}_1 | \cdots | \lambda_n \vec{q}_n\right] \begin{bmatrix} \vec{q}_1 \\ \vdots \\ \vec{q}_n \end{bmatrix} =$$

$$= \boxed{\lambda_1 \underbrace{\vec{q}_1 \vec{q}_1^{\,T}} + \cdots + \lambda_n \underbrace{\vec{q}_n \vec{q}_n^{\,T}} = M} \quad \left(\begin{array}{c}\text{nice} \\ \text{decomposition}\end{array}\right)$$

projection matrices onto the (unitary) axis given by the (orthonormal) eigenvectors.

(*) <u>Proof of</u>: Q square with orthonormal columns $\Rightarrow$ Q has orthonormal rows (and thus, Q is "orthogonal")

$Q^TQ = I$. Then, we know that Q^{-1} exists since

$$\det(Q^TQ) = \det(Q^T)\det(Q) = (\det(Q))^2 = \det(I) = 1 \Rightarrow$$

$$\Rightarrow \det(Q) = \pm 1 \neq 0.$$ (We needed determinants to prove this!)

Therefore, multiplying by Q we obtain

$$QQ^TQ = Q \Rightarrow QQ^TQQ^{-1} = QQ^{-1} \Rightarrow \underbrace{QQ^T = I}_{\Updownarrow\ \text{orthonormal } \underline{\text{rows}}}$$

Before proving the theorem, we need to remember ~~remind~~ some things about complex numbers:

$z \in \mathbb{C} \rightarrow z = a + bi, \ a, b \in \mathbb{R}, \ i^2 = -1$.

- $z_1 + z_2 = (a+bi) + (c+di) \underset{\text{def.}}{=} (a+c) + (b+d)i$.

- $z_1 z_2 = (a+bi)(c+di) \underset{\text{def.}}{=} (ac - bd) + i(ad+bc)$.

- $\bar{z}_1 \underset{\text{def.}}{=} a - bi$ (conjugation)

Some more things we need:

1) $z + \bar{z} = a+bi + a-bi = 2a \in \mathbb{R}$ purely real.

2) $z - \bar{z} = 2ib$ purely imaginary

3) $z = \bar{z} \iff z \in \mathbb{R}$.

4) $z\bar{z} = a^2 + b^2 = |z|^2$ (modul of z)

5) $\vec{u} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \longrightarrow \|\vec{u}\|^2 = \bar{\vec{u}}^T \vec{u} = [\bar{z}_1 \ \bar{z}_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = |z_1|^2 + |z_2|^2$.

6) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Proof of Spectral theorem: Let's split it into two parts.

1) M real and symmetric $\Rightarrow$ real eigenvalues.

Let $\lambda, \vec{u}$ be an eigenvalue and its eigenvector (both maybe complex)

Then (We want to prove λ is real, i.e. $\lambda = \bar{\lambda}$)

$\boxed{1}$ $M\vec{u} = \lambda\vec{u}$

Therefore, taking conjugates, $\overline{M\vec{u}} = \overline{\lambda\vec{u}} \Rightarrow \bar{M}\bar{\vec{u}} = \bar{\lambda}\bar{\vec{u}} \Rightarrow$

$\Rightarrow M\bar{\vec{u}} = \bar{\lambda}\bar{\vec{u}}$ M is real.

$\boxed{2}$ $M\bar{\vec{u}} = \bar{\lambda}\bar{\vec{u}}$

→ Multiply $\boxed{1}$ by $\bar{\vec{u}}^T$: $\bar{\vec{u}}^T M\vec{u} = \bar{\vec{u}}^T\lambda\vec{u} = \lambda\bar{\vec{u}}^T\vec{u} = \lambda\|\vec{u}\|^2$

→ Multiply $\boxed{2}$ by $\vec{u}^T$: $\vec{u}^T M\bar{\vec{u}} = \vec{u}^T\bar{\lambda}\bar{\vec{u}} = \bar{\lambda}\vec{u}^T\bar{\vec{u}} = \bar{\lambda}\|\vec{u}\|^2$ (*)

These are just numbers! (not vectors or matrices)

$[\quad][\quad][\,] = [\quad][\,] =$ number

Therefore,

$\bar{\vec{u}}^T M\vec{u} = (\bar{\vec{u}}^T M\vec{u})^T = \vec{u}^T M^T\bar{\vec{u}} = \vec{u}^T M\bar{\vec{u}}$

same number! M is symmetric

Thus we have found that

$\bar{\vec{u}}^T M \vec{u} = \vec{u}^T M \bar{\vec{u}}$ and we have in (*) that

$$\left.\begin{aligned} \bar{\vec{u}}^T M \vec{u} &= \lambda \|\vec{u}\|^2 \\ \vec{u}^T M \bar{\vec{u}} &= \bar{\lambda} \|\vec{u}\|^2 \end{aligned}\right\}$$

so we conclude that

$$\lambda \|\vec{u}\|^2 = \bar{\lambda} \|\vec{u}\|^2 \Rightarrow (\lambda - \bar{\lambda}) \|\vec{u}\|^2 \underset{\substack{\uparrow \\ \left(\substack{\vec{u} \neq \vec{0} \text{ by} \\ \text{definition of eigenvector}}\right)}}{\Rightarrow} \lambda - \bar{\lambda} = 0 \Rightarrow \underline{\underline{\lambda = \bar{\lambda}}}$$

($\Leftrightarrow$ λ is real)

2) M real, symmetric $\Rightarrow$ orthogonal eigenvectors.

Γ We only prove it for distinct eigenvalues.

But it is always true: if we have an eigenvalue repeated, for symmetric matrices we always obtain a full set of eigenvectors $\underset{\text{Gram-Schmidt}}{\overset{\text{so}}{\rightsquigarrow}}$ orthogonal set $\lrcorner$

Let $\lambda_1 \neq \lambda_2$ eigenvalues with corresponding eigenvectors $\vec{u}_1, \vec{u}_2$.

[Remark: We know from 1) that λ_1 and λ_2 are real (and thus $\vec{u}_1, \vec{u}_2$, being solutions of $(A-\lambda_1 I)\vec{u}_1 = \vec{0}$ ..., are also real).]

$\rightarrow$ We want to show that $\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_1^T \vec{u}_2 = \vec{u}_2^T \vec{u}_1 = 0$:

$$\left.\begin{array}{l} M\vec{u}_1 = \lambda_1 \vec{u}_1 \rightarrow \vec{u}_2^T M \vec{u}_1 = \lambda_1 \vec{u}_2^T \vec{u}_1 \\ M\vec{u}_2 = \lambda_2 \vec{u}_2 \rightarrow \vec{u}_1^T M \vec{u}_2 = \lambda_2 \vec{u}_1^T \vec{u}_2 \\ \text{but } \underset{\substack{\uparrow \\ \text{same number}}}{\vec{u}_2^T M \vec{u}_1} = (\vec{u}_2^T M \vec{u}_1)^T = \vec{u}_1^T M^T \vec{u}_2 \underset{\substack{\uparrow \\ M \text{ symmetric}}}{=} \vec{u}_1^T M \vec{u}_2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lambda_1 \vec{u}_2^T \vec{u}_1 = \lambda_2 \vec{u}_1^T \vec{u}_2 \Rightarrow (\lambda_1 - \lambda_2)\vec{u}_2^T \vec{u}_1 = 0 \underset{\substack{\uparrow \\ \lambda_1 \neq \lambda_2}}{\Rightarrow} \vec{u}_2^T \vec{u}_1 = 0 \,//.$$