$\boxed{\text{Chapter 2}}$: Solving Linear Equations.

I. Algebra and Geometry in Equations.

(Sections 2.1, 2.2)

Example 1:

Let's start with a simple example: Solve for x and y the following systems of equations,

a) $\left.\begin{array}{r} 4x+7y=1 \\ x-2y=4 \end{array}\right\}$ b) $\left.\begin{array}{r} 4x+7y=1 \\ y=-1 \end{array}\right\}$ c) $\left.\begin{array}{r} x=2 \\ y=-1 \end{array}\right\}$

Solution:

Problem c) ✓, b) is also easy: $y=-1 \Rightarrow$

$$\Rightarrow 4x-7=1 \Rightarrow 4x=8 \Rightarrow x=2.$$

But a)?

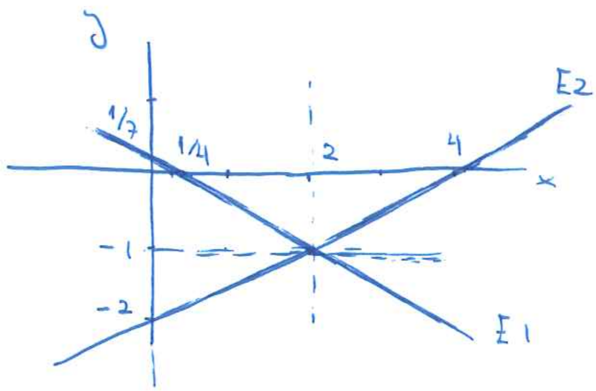
Several ways. We can try solving for x in the second equation first

$x=4+2y$, and substituting that into the first:

$$4(4+2y)+7y=1 \Rightarrow 16+8y+7y=1 \Rightarrow 15y=-15 \Rightarrow y=-1.$$

Or we can solve it geometrically:

Denote | E1 to $4x+7y=1$ (a line)
| E2 to $x-2y=4$ (another line)



E1: $x=0 \rightarrow y=1/7$
$y=0 \rightarrow x=1/4$

E2: $x=0 \Rightarrow y=-2$
$y=0 \rightarrow x=4$

Problem: it can be hard to draw and inaccurate.

II. Gaussian Elimination.

II.1) The Algorithm

We look for a systematic approach.

Consider again
$$\begin{aligned} 4x+7y&=1 \quad E1 \\ x-2y&=4 \quad E2 \end{aligned}$$

We will try to go from this to c), and we will do it passing through b).

- First step: We try to eliminate the dependence on x in the second equation (E2).
  We do that using the "pivot" as a division

$$\text{pivot} \nearrow \left.\begin{array}{ll} \boxed{4}x + 7y = 1 & E1 \\ x - 2y = 4 & E2 \end{array}\right\} \xrightarrow{E2' = E2 - \frac{1}{4}E1} \left.\begin{array}{l} 4x + 7y = 1 \\ -\frac{15}{4}y = \frac{15}{4} \end{array}\right\} \rightarrow$$

$$-\frac{1}{4}\cdot E1 \rightsquigarrow -\frac{1}{4}4x - \frac{7}{4}y = \frac{-1}{4} \qquad -2-\frac{7}{4} = \frac{-15}{4}$$

$$E2 - \frac{1}{4}E1 \rightsquigarrow \cancel{x} - 2y - \cancel{x} - \frac{7}{4}y = 4 - \frac{1}{4} \qquad 4 - \frac{1}{4} = \frac{15}{4}$$

$\rightarrow$ New system
$$\begin{array}{l} 4x + 7y = 1 \\ \qquad\quad y = -1 \end{array} \quad (\text{Triangular})$$

- Second step: Back substitution $y = -1$ into E1,

$$4x - 7 = 1 \rightsquigarrow x = 2$$

<u>Method</u>: Algorith for Gaussian Elimination

$(n = 1)$

$\rightarrow$ 1) Rearrange your equations so that the n-th variable in the n-th equation has a <u>non-zero</u> coefficient (called PIVOT).

$n \leftarrow n+1$

2) Use the PIVOT to clear out the n-th variable from all equations below the n-th one.

3) Is the system triangular? $\xrightarrow{\text{Yes}}$ Back substitution.

NO

Example: Solve for x, y, z in the following system

$$\left.\begin{array}{ll} E1 & x+y+z=4 \\ E2 & x-2y-2z=7 \\ E3 & 2x-3y+3z=-5 \end{array}\right\}$$

Q: Geometric interpretation of E1?

Solution:

Try it geometrically → intersection of planes → hard!

Try it ~~substitu~~ solving for z first and substituting ...
doable but kind of messy

Let's use "our" algorithm (gaussian elimination):

n = 1

1) Our 1-st variable (x) in the 1-st equation has coefficient 1 (NON-ZERO), so the pivot is 1.

2) Rewrite E2 and E3 without x:

$$E2' = E2 - E1 \longrightarrow 0 - 3y - 3z = 3$$

$$E3' = E3 - 2 \cdot E1 \longrightarrow 0 - 5y + z = -13$$

New system:

$$\left.\begin{array}{r} x+y+z=4 \\ -3y-3z=3 \\ -5y+z=-13 \end{array}\right\} \text{Not triangular yet.}$$

(No x)

Q: ask for pivot Multiplier: $\dfrac{\text{entry to eliminate}}{\text{pivot}}$

<u>$n=2$</u>: The pivot is now -3 $(E2' \equiv -3y - 3z = 3)$

So $E3'' = E3' - \boxed{\dfrac{-5}{-3}} E2$, i.e.,

$$\left.\dfrac{-5}{-3} E2 \equiv \dfrac{5}{3} E2 \equiv -5y - 5z = 5 \;\right\} \to E3'' \equiv 0 + 6z = -18$$

New system:

$$\left.\begin{array}{rl} x + y + z = 4 & E1 \\ -3y - 3z = 3 & E2' \\ 6z = -18 & E3'' \end{array}\right\} \text{Triangular} \to$$

pivots

→ Easy to solve backwards:

First, use $E3''$ to get $z = -3$.

Second, substitute this in $E2'$ to get $-3y + 9 = 3 \Rightarrow y = 2$.

Third, substitute both z and y in $E1$ to get $x = 5$.

So, the solution is $x = 5$, $y = 2$, $z = -3$.

(only in one point the three plane intersect!).

• <u>Question</u>: Can this algorithm go wrong?

Q

↳ What happens if we find a ZERO instead of a pivot?

II. 2) Breakdown of Elimination: Singular cases.

Let's see what happens in the following 3 systems of linear equations:

Case 1	Case 2	Case 3
$\left.\begin{array}{l} x+y+z=4 \\ 2x+2y-2z=7 \\ 2x-y-z=11 \end{array}\right\}$	$\left.\begin{array}{l} x+y+z=4 \\ x-2y-2z=7 \\ 2x-y-z=10 \end{array}\right\}$	$\left.\begin{array}{l} x+y+z=4 \\ x-2y-2z=7 \\ 2x-y-z=11 \end{array}\right\}$

- Case 1

Pivot $\to 1$, so

$$\begin{array}{rl} x+y+z=4 & E1 \\ -4z=-1 & E2' \\ -3y-3z=3 & E3' \end{array}$$

Problem! Zero y-coefficient $\to$ no pivot

But, our algorithm solved this: step 1) Rearrange!

$$\left.\begin{array}{r} x+y+z=4 \\ -3y-3z=3 \\ -4z=-1 \end{array}\right\} \quad \text{(Backwards...)}$$

Remark: This is NOT a singular case. The algorithm works and we have one and only one solution.

• <u>Case 2</u>

After 1st iteration we obtain

$$\begin{aligned} x + y + z &= 4 \quad E1 \\ -3y - 3z &= 3 \quad E2' \\ -3y - 3z &= 2 \quad E3' \end{aligned}$$

Second iteration (!):

$$\begin{aligned} x + y + z &= 4 \quad E1 \\ -3y - 3z &= 3 \quad E2' \end{aligned}$$

Not possible! $\longrightarrow$ $0 = -1 \quad E3''$

↳ Our system has <u>no</u> solution : Singular case with zero solutions.

• <u>Case 3</u>

We get

$$\begin{aligned} x + y + z &= 4 \quad E1 \\ -3y - 3z &= 3 \quad E2' \\ -3y - 3z &= 3 \quad E3' \end{aligned} \quad \longrightarrow \quad \begin{aligned} x + y + z &= 4 \quad E1 \\ -3y - 3z &= 3 \quad E2' \\ 0 &= 0 \quad E3'' \end{aligned}$$

Always true!

So our system is:

$$\left.\begin{aligned} x + y + z &= 4 \\ -3y - 3z &= 3 \end{aligned}\right\}$$ Less equations than ~~variables~~ unknowns $\Rightarrow$

$\Rightarrow$ Infinite solutions (singular case).

<u>Remark</u>: In the singular case with infinite solutions, we might want to find one or two particular solutions, or better, the whole set of solutions.

How?

Since we have more variables than pivots, these extra variables act as parameters.

In our Case 3: Let $z \in \mathbb{R}$ be a parameter (or "free variable").

Then,

$$-3y = 3 + 3z \Rightarrow y = -1 - z,$$

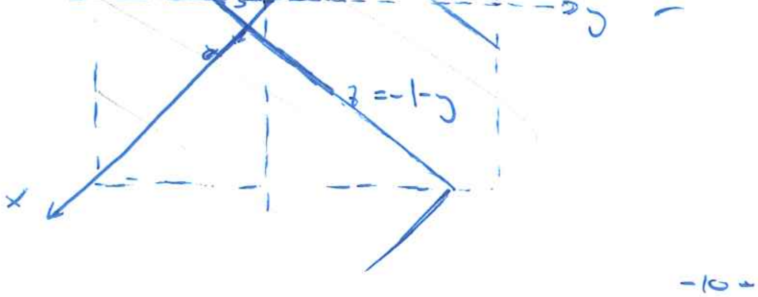
and

$$x + (-1-z) + z = 4 \Rightarrow x = 5.$$

So our set of solutions are: $\begin{matrix} x = 5 \\ y = -1 - z \end{matrix}$ with $z \in \mathbb{R}$.

Geometrically, this is a line in the plane $x = 5$.

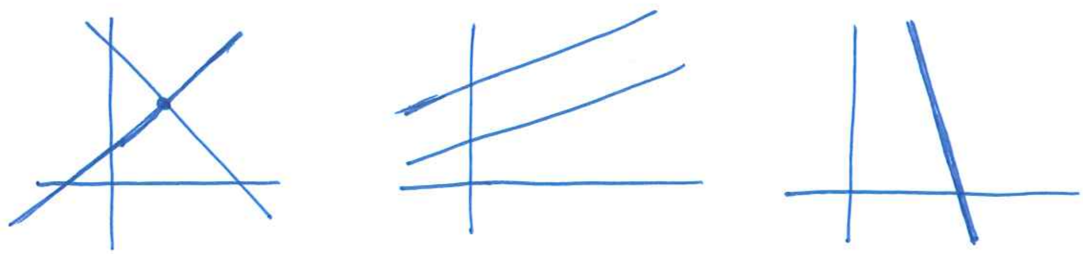
$y = -1 - z \Leftrightarrow z = -1 - y$



Important! In linear systems of equations, only these three possibilities: $\{0, 1, \infty\}$ solutions

Why?

We can see it in the 2 by 2 case thinking geometrically



Singular cases.

Can you think why in the n-dimensional case?
It will be easier after next lecture (algebra viewpoint).