MATH 312 LECTURE 10	: Linear Transformations.

I.) Introduction.

Def: Linear Transformation.

Let $\mathcal{U}, V$ be two vector spaces. A linear transformation from $\mathcal{U}$ to V is a function

$$\left.\begin{array}{l} T : \mathcal{U} \longrightarrow V \\ \quad \vec{u} \longmapsto \vec{v} = T(\vec{u}) \end{array}\right\} \text{ that satisfies two properties:}$$

1) $T(\vec{u}_1 + \vec{u}_2) = T(\vec{u}_1) + T(\vec{u}_2)$, for all $\vec{u}_1, \vec{u}_2 \in \mathcal{U}$,

2) $T(\lambda \vec{u}) = \lambda T(\vec{u})$. $\lambda \in \mathbb{R}$ (scalar).

Remarks:
- $T(\vec{0}) = \vec{0}$
- $T(\alpha \vec{u}_1 + \beta \vec{u}_2) = \alpha T(\vec{u}_1) + \beta T(\vec{u}_2)$.

Example: Which are linear transf.? $(a, b \in \mathbb{R})$

a) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ a \end{bmatrix}$, ✓ b) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ab$, ×

c) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b \\ 1 \end{bmatrix}$, × d) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a - b$. ✓

<u>Remark</u>: Both a) and d) can be written using a matrix:

a) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, d) $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = [1 \quad -1]\begin{bmatrix} a \\ b \end{bmatrix}$.

<u>Example</u>: Which are linear?

1) $\left.\begin{array}{l} T : \mathbb{R}^3 \to \mathbb{R}^3 \\ \vec{u} \mapsto T(\vec{u}) = u_1 + u_2 + u_3 \end{array}\right\} \checkmark$

2) $\left.\begin{array}{l} T : \mathbb{R}^3 \to \mathbb{R}^3 \\ \vec{u} \mapsto \|\vec{u}\| \end{array}\right\} \times$

3) $\left.\begin{array}{l} T : \mathcal{P}_2 \to \mathcal{P}_3 \\ p \mapsto T(p) \end{array}\right\}, \quad \checkmark$

where $T(p)(x) = (1+x)p(x)$.

<u>Example</u>: (Linearity)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be linear. Assume we know that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = -1, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = 2.$$

What is $T\left(\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}\right)$? And $T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$?

- Remark: The previous example shows that if we know how the basis vectors are transformed, then we know everything!

Let $\mathcal{U}$, V vector spaces, $B_u = \{\vec{u}_1, \dots, \vec{u}_n\}$ basis of $\mathcal{U}$.

Assume we know $T(\vec{u}_1), \dots, T(\vec{u}_n)$.

Then, we know $T(\vec{x})$ for all $\vec{x} \in \mathcal{U}$. Indeed,

$\vec{x} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n$ (since $\{\vec{u}_1, \dots, \vec{u}_n\}$ basis for $\mathcal{U}$).

Due to linearity properties,

$T(\vec{x}) = c_1 T(\vec{u}_1) + \dots + c_n T(\vec{u}_n)$ //.

II.5 The matrix of a linear transformation.

Example: The derivative

Let $U = P_2 \equiv$ vector space of polyn. degree ≤ 2.

$V = P_1 \equiv$ " " " " " ≤ 1.

We can choose basis for U and V: $B_u = \{1, x, x^2\}$

$B_v = \{1, x\}$

Consider the following linear transformation:

$$\left.\begin{aligned} T : U &\to U \\ p(x) &\mapsto T(p(x)) = p'(x) \end{aligned}\right\} \text{(derivative)}.$$

That is, if we denote $p(x) = a + bx + cx^2$, then

$T(p(x)) = b + 2cx$.

→ In our basis B_u, this is $p|_{B_u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $T(p)|_{B_u} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix}$.

Then, we can write our linear transformation as follows:

$$T(p)|_{B_u} = \begin{bmatrix} b \\ 2c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Indeed, $p(x) = 4x^2+3 \Rightarrow T\left(\begin{bmatrix}3\\0\\4\end{bmatrix}\right)\Big|_{B_u} = \begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}\begin{bmatrix}3\\0\\4\end{bmatrix} = \begin{bmatrix}0\\8\\0\end{bmatrix} \Rightarrow$

$\Rightarrow T(p(x)) = 8x$.

• Def: Kernel of a lin. transformation.

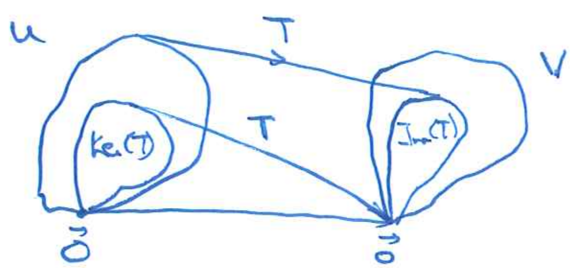
Let $\left.\begin{aligned} T: \mathcal{U} &\to V \\ \vec{u} &\mapsto \vec{v} = T(\vec{u}) \end{aligned}\right\}$ be a lin. transf.

Its kernel is the vector space given by

$\ker(T) = \{ \vec{u} \in \mathcal{U} : T(\vec{u}) = \vec{0} \}$.

• Def: Range of lin. transf.

$\operatorname{Im}(T) = \{ \vec{v} \in V : \exists \vec{u} \in \mathcal{U} \text{ with } T(\vec{u}) = \vec{v} \}$.



→ So, if the linear transf. is given by a matrix, $T(\vec{x}) = A\vec{x}$, clearly we have that

$$\left.\begin{aligned}\ker(T) &\equiv N(A) \\ \operatorname{Im}(T) &\equiv C(A)\end{aligned}\right\|$$

- For the derivative example, what is the kernel?

We had: $T(p(x)) = p'(x)$

$$T(p)|_{B_u} = M\, p|_{B_u}, \text{ with } M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

It's clear that:

$T(p(x)) = p'(x) = 0 \iff p(x) = a \in \mathbb{R}$ (constant).

We can also check that:

$N(M) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, i.e., constant polynomials.

- What is the range?

For a general $p(x) = a + bx + cx^2 \Rightarrow p'(x) = b + 2cx$.

Notice that $T(p(x)) = b + 2cx$ is always of degree at most 1. That is,

$\text{Im}(T) \equiv$ polynomials of degree ≤ 1.

Indeed, we can see that

$C(M) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$.

Remark: Notice that if we write the previous lin. transf. as

$$\left.\begin{array}{l} T: U \longrightarrow V \\ p(x) \longmapsto T(p(x)) = p'(x) \end{array}\right\},$$ using basis B_U, B_V,

its matrix representation changes. Indeed:

$$p(x) = a + bx + cx^2 \longrightarrow T(p(x)) = b + 2cx$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\underbrace{p|_{B_U} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad\qquad T(p)|_{B_V} = \begin{bmatrix} b \\ 2c \end{bmatrix}}$$

$$\Rightarrow T(p)|_{B_V} = \begin{bmatrix} b \\ 2c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \tilde{M}\, p|_{B_U}$$

Still, one can check that

$$N(\tilde{M}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \rightsquigarrow \operatorname{Ker}(T) \equiv \text{constants}.$$

$$C(\tilde{M}) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} \rightsquigarrow \operatorname{Im}(T) \equiv \text{polynomials of degree} \leq 1.$$

→ Moral: The matrix of a lin. transf. depends on the basis we use for the input and output spaces ||.

• **Question:** Given a general lin. transformation T, can it always be written using a matrix?

↳ **Yes.**

Let's see how:

• **Matrix of a lin. transf.:** Let $T: U \to V$ be a lin. transf. and $\mathcal{U} = \{\vec{u}_1, \dots, \vec{u}_n\}$, $\mathcal{V} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be basis of U, V. Then,

$$T(\vec{u})|_{\mathcal{V}} = M_{\mathcal{U}}^{\mathcal{V}} \, \vec{u}|_{\mathcal{U}}, \quad \text{where}$$

$$M_{\mathcal{U}}^{\mathcal{V}} = \left[T(\vec{u}_1)|_{\mathcal{V}} \;\middle|\; T(\vec{u}_2)|_{\mathcal{V}} \;\middle|\; \cdots \;\middle|\; T(\vec{u}_n)|_{\mathcal{V}} \right]$$

Remark: $M_{\mathcal{U}}^{\mathcal{V}}$ is an $m \times n$ matrix $\left(\text{where } \begin{matrix} m = \dim V \\ n = \dim U \end{matrix}\right)$.

• Let's see why:

Take $\vec{u} \in U$. We can write in the basis $\mathcal{U}$:

$\vec{u} = \alpha_1 \vec{u}_1 + \dots + \alpha_n \vec{u}_n$. That is, $\vec{u}|_{\mathcal{U}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$.

Then, $\vec{u}$ is transformed by T using linearity properties:

$$T(\vec{u}) = \alpha_1 T(\vec{u}_1) + \dots + \alpha_n T(\vec{u}_n) =$$

$$= \left[T(\vec{u}_1) \Big| \dots \Big| T(\vec{u}_n) \right] \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\vec{u}|_{\mathcal{U}}}$$

Note that both $T(\vec{u})$ and $T(\vec{u}_1), \dots T(\vec{u}_n)$ can be written in any basis of V. If we choose $\mathcal{V}$:

$$T(\vec{u})\big|_{\mathcal{V}} = \left[T(\vec{u}_1)\big|_{\mathcal{V}} \Big| \dots \Big| T(\vec{u}_n)\big|_{\mathcal{V}} \right] \vec{u}\big|_{\mathcal{U}}$$

• <u>Exercise</u>: Let $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be basis of two vector spaces U, V. Assume that

$$\left.\begin{array}{l} T(\vec{u}_1) = \vec{v}_1 \\ T(\vec{u}_2) = T(\vec{u}_3) = \vec{v}_1 + \vec{v}_3 \end{array}\right\} \text{ and that } T \text{ is linear.}$$

1) Find the matrix that represents this lin. transf. in the basis $\mathcal{U}$ to $\mathcal{V}$, i.e., $M_{\mathcal{U}}^{\mathcal{V}}$.

2) Find $T(\vec{u}_1 + \vec{u}_2 + \vec{u}_3)$.

Sol:

1) $M_{\mathcal{U}}^{\mathcal{V}} = \begin{bmatrix} T(\vec{u}_1)|_{\mathcal{V}} & T(\vec{u}_2)|_{\mathcal{V}} & T(\vec{u}_3)|_{\mathcal{V}} \end{bmatrix}$, and

$$T(\vec{u}_1)|_{\mathcal{V}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ T(\vec{u}_2)|_{\mathcal{V}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = T(\vec{u}_3)|_{\mathcal{V}} \rightsquigarrow M_{\mathcal{U}}^{\mathcal{V}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

2) Two ways:

→ Direct: $T(\vec{u}_1 + \vec{u}_2 + \vec{u}_3) = T(\vec{u}_1) + T(\vec{u}_2) + T(\vec{u}_3) =$

$$= \vec{v}_1 + 2(\vec{v}_1 + \vec{v}_3) = 3\vec{v}_1 + 2\vec{v}_3 \,/\!/.$$

→ Using the matrix:

$$T(\vec{u}_1 + \vec{u}_2 + \vec{u}_3)|_{\mathcal{V}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} /\!/.$$

• <u>Exercise:</u> Let $T : P_3 \to \mathbb{R}^4$ be defined by

$$T(a_1 + a_2 x + a_3 x^2 + a_4 x^3) = \begin{bmatrix} a_4 + a_3 \\ 2a_3 \\ a_2 \\ \frac{1}{2} a_1 \end{bmatrix}.$$

Consider the standard basis for both P_3 and $\mathbb{R}^4$.

1) Find the matrix of T (using the standard basis).

2) Use the matrix to find $T(2x^3 - 4x + 3)$.

<u>Remark:</u> Part 2) is useful as a check of 1), since it's easy to see that

$$T(2x^3 - 4x + 3) = \begin{bmatrix} 2 \\ 0 \\ -4 \\ 3/2 \end{bmatrix}$$

$a_1 = 3$
$a_2 = -4$
$a_3 = 0$, $a_4 = 2$

<u>Sol:</u> $M = \left[T(1) \mid T(x) \mid T(x^2) \mid T(x^3) \right]$, [I should write $T(1)|_E$, $T(x)|_E$ ... to be more precise]

$$T(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}, \quad T(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad T(x^2) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad T(x^3) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So $M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \\ 3/2 \end{bmatrix}$ //.

• **Exercise:** Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ -2x_3 \end{bmatrix}.$$

Find the matrix using bases $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \right\}, \mathcal{V} = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}.$

Solution:

→ First way: $M_{\mathcal{U}}^{\mathcal{V}} = \Big[T(\vec{u}_1)|_{\mathcal{V}} \quad T(\vec{u}_2)|_{\mathcal{V}} \quad T(\vec{u}_3)|_{\mathcal{V}} \Big],$

$$T(\vec{u}_1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad T(\vec{u}_2) = \begin{bmatrix} -3 \\ -4 \end{bmatrix}, \quad T(\vec{u}_3) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

We have to write these in the basis $\mathcal{V}$:

$$M_{\mathcal{V}}\, T(\vec{u}_1)|_{\mathcal{V}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \equiv T(\vec{u}_1)|_{\mathcal{E}}, \quad M_{\mathcal{V}}\, T(\vec{u}_2)|_{\mathcal{V}} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \ldots$$

Gauss-Jordan:

$$\left[\begin{array}{cc|ccc} 3 & 2 & 1 & -3 & 5 \\ -1 & 5 & -2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|ccc} 1 & 0 & 9/17 & -7/17 & 25/17 \\ 0 & 1 & -5/17 & -15/17 & 5/17 \end{array}\right]$$

$\underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}}_{M_{\mathcal{V}}}$, $\quad \underbrace{\begin{array}{ccc} \overbrace{9/17}^{T(\vec{u}_1)|_{\mathcal{V}}} & -7/17 \;\ldots & 25/17 \\ -5/17 & -15/17 & 5/17 \end{array}}_{M_{\mathcal{U}}^{\mathcal{V}}}$

→ Second way: Using the standard basis for $\mathbb{R}^3$ and $\mathbb{R}^2$,

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \text{ i.e., } T(\vec{x})\big|_{\mathcal{E}} = M\,\vec{x}\big|_{\mathcal{E}} .$$

Now we can use what we know about changing of basis:

- We want to obtain: $T(\vec{x})\big|_{\mathcal{V}} = M^{\mathcal{V}}_{\mathcal{U}}\,\vec{x}\big|_{\mathcal{U}}$,

and we know that

$$\left.\begin{aligned} M_{\mathcal{U}}\,\vec{x}\big|_{\mathcal{U}} &= \vec{x}\big|_{\mathcal{E}} \\ M_{\mathcal{V}}\,T(\vec{x})\big|_{\mathcal{V}} &= T(\vec{x})\big|_{\mathcal{E}} \end{aligned}\right\} \quad \left(\begin{aligned} \text{where } M_{\mathcal{U}} &= \left[\vec{u}_1 \,\middle|\, \vec{u}_2 \,\middle|\, \vec{u}_3\right] \\ M_{\mathcal{V}} &= \left[\vec{v}_1 \,\middle|\, \vec{v}_2\right] \end{aligned} \right)$$

Thus,

$$T(\vec{x})\big|_{\mathcal{E}} = M\,\vec{x}\big|_{\mathcal{E}} \Rightarrow M_{\mathcal{V}}\,T(\vec{x})\big|_{\mathcal{V}} = M\,M_{\mathcal{U}}\,\vec{x}\big|_{\mathcal{U}} \Rightarrow$$

$$\Rightarrow \Big\| \; T(\vec{x})\big|_{\mathcal{V}} = M_{\mathcal{V}}^{-1}\,M\,M_{\mathcal{U}}\,\vec{x}\big|_{\mathcal{U}}$$

Let's check it:

$$MM_{\mathcal{U}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 5 \\ -2 & -4 & 0 \end{bmatrix} /\!/ .$$

• <u>Remark</u>: Note that $M_{\mathcal{U}}, M_{\mathcal{V}}$ are indeed the change of basis matrix from $\mathcal{U}$ to $\mathcal{E}$, $\mathcal{V}$ to $\mathcal{E}$, i.e.,

$M_{\mathcal{U}} \equiv M_{\mathcal{U}}^{\mathcal{E}}$, $M_{\mathcal{V}} \equiv M_{\mathcal{V}}^{\mathcal{E}}$.

So we can generalise what we saw in the "second way":

Change of basis and the matrix of a lin. transf.

Let $T: U \to V$ be a lin. transformation, and let $\mathcal{U}_1, \mathcal{U}_2$, $\mathcal{V}_1, \mathcal{V}_2$ be basis of U, V (respect.).

Then, if $T(\vec{u})|_{\mathcal{V}_1} = M_{\mathcal{U}_1}^{\mathcal{V}_1} \vec{u}|_{\mathcal{U}_1}$,

$T(\vec{u})|_{\mathcal{V}_2} = M_{\mathcal{U}_2}^{\mathcal{V}_2} \vec{u}|_{\mathcal{U}_2}$,

it holds that:

$$\boxed{M_{\mathcal{U}_2}^{\mathcal{V}_2} = P_{\mathcal{V}_2 \leftarrow \mathcal{V}_1} M_{\mathcal{U}_1}^{\mathcal{V}_1} P_{\mathcal{U}_1 \leftarrow \mathcal{U}_2}}$$

• <u>Proof</u>:

$$T(\vec{u})|_{\mathcal{V}_2} = P_{\mathcal{V}_2 \leftarrow \mathcal{V}_1} T(\vec{u})|_{\mathcal{V}_1} = P_{\mathcal{V}_2 \leftarrow \mathcal{V}_1} M_{\mathcal{U}_1}^{\mathcal{V}_1} \vec{u}|_{\mathcal{U}_1} = P_{\mathcal{V}_2 \leftarrow \mathcal{V}_1} M_{\mathcal{U}_1}^{\mathcal{V}_1} P_{\mathcal{U}_1 \leftarrow \mathcal{U}_2} \vec{u}|_{\mathcal{U}_2} //.$$