MATH 312 LECTURE 12	: Projections and Least-Squares.

Last day: orthogonal complements, FTLA.

• Exercise: Let $V = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$.

Find a basis for V and $V^{\perp}$.

Solution:

A basis for V can be found by solving the equation $x_1 + x_2 + x_3 = 0$:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow \left. \begin{aligned} x_2 &= \alpha \\ x_3 &= \beta \\ x_1 &= -\alpha - \beta \end{aligned} \right\} \Rightarrow \vec{x} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

free variables

So, basis for $V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Remark: V was a plane in $\mathbb{R}^3 \rightarrow \dim V = 2$

• $V^{\perp}$? By definition, $\vec{y} \in V^{\perp}$ if $\vec{y} \cdot \vec{x} = 0$ for all $\vec{x} \in V$.

Let's impose this condition:

$$\vec{x} \in V \rightarrow \vec{x} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad (\alpha, \beta \in \mathbb{R})$$

$$\vec{y} \in V^{\perp} \Leftrightarrow \vec{y} \cdot \left(\alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) = 0 \overset{\forall \alpha, \beta \in \mathbb{R}}{\Leftrightarrow}$$

$$\Leftrightarrow \alpha \, \vec{y} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \, \vec{y} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0 \overset{\forall \alpha, \beta \in \mathbb{R}}{\underset{(*)}{\Leftrightarrow}} \left. \begin{array}{l} -y_1 + y_2 = 0 \\ -y_1 + y_3 = 0 \end{array} \right\}$$

Remark: (*) We have just seen that ~~...~~ for a vector $\vec{y}$ to be in $V^{\perp}$, it is enough that $\vec{y}$ is perpendicular to all the basis vectors of V.

Thus, $\vec{y} \in V^{\perp} \Leftrightarrow \left. \begin{array}{l} -y_1 + y_2 = 0 \\ -y_1 + y_3 = 0 \end{array} \right\} \Leftrightarrow$

$$\Leftrightarrow \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & +1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

free variable.

Let's $y_3 = \alpha \rightarrow y_1 = \alpha$, $y_2 = \alpha$,

So $\vec{y} \in V^{\perp}$ if $\vec{y} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow$ basis for $V^{\perp}$ $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Remark: V is a plane in $\mathbb{R}^3$, so we knew $V^\perp$ is a line orthogonal to that plane.

We could obtain the basis $\begin{bmatrix}1\\1\\1\end{bmatrix}$ immediately:

$$\vec{x} \in V: \quad x_1 + x_2 + x_3 = 0 \rightsquigarrow \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \rightsquigarrow$$

$\rightsquigarrow$ That is, $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is perpendicular to any $\vec{x} \in V$!

Since we knew that $\dim V = 2$, then $\dim V^\perp = 3 - 2 = 1$,

so $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is a basis for $V^\perp$!

• Exercise: Find a matrix with the following properties, or explain why it is not possible.

a) $A\vec{x} = \begin{bmatrix}0\\1\\1\end{bmatrix}$ has a solution and $A^T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

b) $A\vec{x} = \begin{bmatrix}0\\1\\1\end{bmatrix}$ has a solution and $A^T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

c) $A\vec{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ has a solution and the rows of A add up to a row of zeros.

Solution:

a) $A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ has a solution $\Rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in C(A)$

Let's try with

$$A = \begin{bmatrix} 0 & a & d \\ 1 & b & e \\ 1 & c & f \end{bmatrix} ; \quad \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & d \\ 1 & b & e \\ 1 & c & f \end{bmatrix} = \begin{bmatrix} 0 & a & d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow a = d = 0.$$

So, for example, $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ satisfies the conditions.

b) We can try as in a):

$$A = \begin{bmatrix} 0 & a & d \\ 1 & b & e \\ 1 & c & f \end{bmatrix} ; \quad \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & d \\ 1 & b & e \\ 1 & c & f \end{bmatrix} = \begin{bmatrix} 1 & b & e \end{bmatrix} \underset{\text{impossible}}{=} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} !$$

Indeed, we could see this from the start:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in C(A), \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in N(A^T) \quad \text{but} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \neq 0 \ !!$$

(recall that $C(A) \perp N(A^T)$).

c) Notice that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in C(A), \text{ but also } \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$\parallel$ (rows add up to zero)

$$= \begin{bmatrix} (a_{11} + a_{21} + a_{31}) & \cdots & (a_{1n} + a_{2n} + a_{3n}) \end{bmatrix}$$

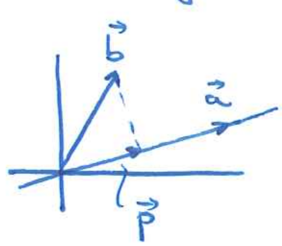
Thus, we need $\begin{bmatrix}1\\1\\1\end{bmatrix} \in C(A)$ and $\in N(A^T)$, which is impossible since $C(A) \perp N(A^T)$.

Indeed, $\begin{bmatrix}1\\1\\1\end{bmatrix} \in C(A) \rightarrow \begin{bmatrix}1\\1\\1\end{bmatrix} = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \dots + \alpha_n \vec{a}_n$, where $\vec{a}_1, \dots, \vec{a}_n$ are the columns of A.

But $\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \vec{a}_i = 0$ for all $i = 1, \dots, n$! $\qquad \perp$.

II.) Projections.

II.1) Projection onto a line:



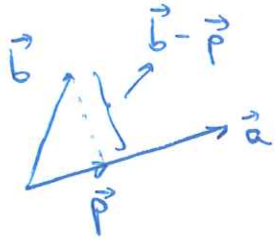
$V \equiv$ subspace spanned by $\vec{a}$ (a line)

- Given $\vec{b}$ and $\vec{a}$, how to find $\vec{p} \equiv$ projection of $\vec{b}$ onto $\vec{a}$?

<u>Remark</u>: we will also use the notation $\vec{p} \equiv P_V \vec{b}$ $\|$.

<u>Answer</u>: Well, $\vec{p}$ is on the line, so it has to be some multiple of $\vec{a}$: $\quad \vec{p} = \lambda \vec{a}$, where we need to find $\lambda \in \mathbb{R}$.

• How?



We need to use that $\vec{b}-\vec{p}$ is perpendicular to $\vec{a}$:

$$\left.\begin{array}{l}\vec{a}\cdot(\vec{b}-\vec{p})=0\\ \vec{p}=\lambda\vec{a}\end{array}\right\}\Rightarrow$$

$$\Rightarrow \vec{a}\cdot(\vec{b}-\lambda\vec{a})=0 \Rightarrow \vec{a}\cdot\vec{b}=\lambda\,\vec{a}\cdot\vec{a} \Rightarrow \lambda=\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}.$$

• In summary,

$$\boxed{P_V\,\vec{b}\equiv\vec{p}=\left(\frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\right)\vec{a}=\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\right)\vec{a}}$$

• <u>Remark</u>: Using matrix notation, we can write it as follows:

$$\vec{p}=\vec{a}\,\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}=\vec{a}\,\frac{\vec{a}^T\vec{b}}{\|\vec{a}\|^2}=\frac{\vec{a}\,\vec{a}^T}{\|\vec{a}\|^2}\,\vec{b}=P\vec{b}, \text{ where}$$

$$\boxed{P=\frac{\vec{a}\,\vec{a}^T}{\|\vec{a}\|^2}=\frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}}$$ is the Projection matrix (onto $\vec{a}$).

$\Rightarrow$ So the projection of vectors onto a line is a linear transformation.

• **Example:** Find the projection of $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ onto the line given by

$$\left.\begin{array}{l} x - y = 0 \\ x - z = 0 \end{array}\right\}.$$

Sol:

We first need a basis for the line (our vector $\vec{a}$).

$$\left.\begin{array}{l} x - y = 0 \\ x - z = 0 \end{array}\right\} \rightsquigarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \text{ Therefore,}$$

$$\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \vec{a} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

• Q: What is the projection matrix?

$$P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T \vec{a}} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Check: $\vec{p} = P\vec{b} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

• Q: What is $N(P)$? $C(P)$? P^2?

$C(P) \to$ The line! (of course, all vectors get projected onto that line when multiplied by P).

$N(P) \to$ The orthogonal complement of the line!

$\downarrow$

Indeed, everything that is perpendicular to the line is projected to zero.
We can check it in the example:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \left. \begin{array}{l} y = \alpha \\ z = \beta \\ x = -\alpha - \beta \end{array} \right\}$$

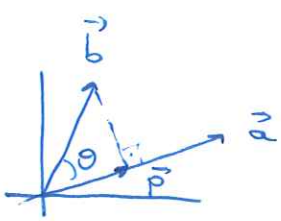
$$N(P) = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \equiv \text{plane orthogonal to the line } \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \checkmark .$$

- Finally, $P^2 = P$!

$$\hookrightarrow \frac{1}{3^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3^2} \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

This is true for all projections matrices: of course, because $P\vec{b}$ is already on the line, so $P(P\vec{b})$ is the same as $P\vec{b}$.

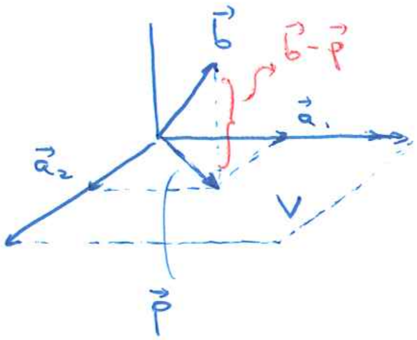
- <u>Exercise</u> (Home): Find projection of $\vec{b}$ onto $\vec{a}$ using trigonometry:



$$\left. \begin{array}{l} \cos\theta = \dfrac{\|\vec{p}\|}{\|\vec{b}\|} \\ \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta \end{array} \right\} \to \|\vec{p}\|, \vec{p} \,?$$

II. 2) Projection onto a subspace.

Consider first a plane. Choose a basis for the plane, $\{\vec{a}_1, \vec{a}_2\}$ in $\mathbb{R}^n$.



- Once we know $\vec{a}_1, \vec{a}_2$, given $\vec{b}$, what is its projection onto V?

  ↳ I.e., find $\vec{p} \equiv P_V \vec{b}$.

→ Since $\vec{p} \in V$, it has to be a linear combination of $\vec{a}_1, \vec{a}_2$:

$$\vec{p} = \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 = \underbrace{\left[\begin{array}{c|c} \vec{a}_1 & \vec{a}_2 \end{array}\right]}_{A} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\vec{\alpha}} \equiv A\vec{\alpha}.$$

→ Let's use that $\vec{b}-\vec{p}$ is orthogonal to the plane:
(so in particular $\perp$ to $\vec{a}_1$ and $\vec{a}_2$)

$$\vec{a}_1 \cdot (\vec{b}-\vec{p}) = 0 \rightarrow \vec{a}_1^T(\vec{b}-A\vec{\alpha}) = 0 \rightarrow \begin{bmatrix} \vec{a}_1^T \end{bmatrix} \begin{bmatrix} \vec{b}-A\vec{\alpha} \end{bmatrix} = 0$$

$$\vec{a}_2 \cdot (\vec{b}-\vec{p}) = 0 \rightarrow \vec{a}_2^T(\vec{b}-A\vec{\alpha}) = 0 \hookrightarrow \begin{bmatrix} \vec{a}_2 \end{bmatrix} \begin{bmatrix} \vec{b}-A\vec{\alpha} \end{bmatrix} = 0 \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \vec{a}_1^T \\ \vec{a}_2^T \end{bmatrix} \begin{bmatrix} \vec{b}-A\vec{\alpha} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow A^T(\vec{b}-A\vec{\alpha}) = 0 \Rightarrow$$

$\Rightarrow \left\| \; A^T A \vec{\alpha} = A^T \vec{b} \right.$ (we need to solve for $\vec{\alpha}$).

- Assuming A^TA has an inverse (*), we have that

  $\vec{\alpha} = (A^TA)^{-1} A^T \vec{b}$, therefore $\vec{p} = A\vec{\alpha} = A(A^TA)^{-1}A^T\vec{b}$.

- In summary, given a plane V in $\mathbb{R}^3$, with basis $\{\vec{a}_1, \vec{a}_2\}$, and a vector $\vec{b} \in \mathbb{R}^3$, the projection of $\vec{b}$ onto V is

$$\left\| \; \vec{p} = P_V \vec{b} = \underbrace{A(A^TA)^{-1}A^T}\vec{b} \; \right\|$$

$$P = A(A^TA)^{-1}A^T, \text{ with } A = \left[\vec{a}_1 \;\vdots\; \vec{a}_2\right]$$

Projection matrix onto V.

(*) A^TA is always square an symmetric.

$\hookrightarrow$ It has an inverse if and only if the columns of A are l.i.

<u>Remark</u>: For these problems, since ~~A has~~ the columns of A are the vectors in a basis of V, A^TA is invertible.

Example: 1) Find the projection matrix P onto the plane $x+y+z=0$, and the projection of $\vec{b}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ onto the plane.

2) Find $N(P)$.

3) Let V be the plane in a). Using the basis $\mathcal{V}=\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$, find the matrix of the following linear transformation: $\left\{\begin{array}{l} T:\mathbb{R}^3\to V \\ \vec{b}\mapsto T(\vec{b})\big|_{\mathcal{V}}=P_V\vec{b}\,\big|_{\mathcal{V}}\end{array}\right\}$.

Solution:

1) We need a basis for $V \to \left\{\underbrace{\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{\vec{a}_1},\underbrace{\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{\vec{a}_2}\right\}$.

We construct A:

$$A=\begin{bmatrix}1&1\\-1&0\\0&-1\end{bmatrix}$$

Finally, $P=A(A^TA)^{-1}A^T$, $A^TA=\begin{bmatrix}1&-1&0\\0&1&-1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$,

$(A^TA)^{-1}=\frac{1}{3}\begin{bmatrix}2&1\\1&2\end{bmatrix}$. Thus,

$$P=\frac{1}{3}\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&-1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}\begin{bmatrix}2&-1&-1\\1&1&-2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$$

- $P_V \vec{b} = P\vec{b} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$

2) $N(P)$: all vectors $\vec{x} \in \mathbb{R}^3$ such that $P\vec{x} = \vec{0}$.

Since P is the matrix that projects onto the plane V, we see that $N(P) = V^\perp$, that is, the vectors on the line perpendicular to the plane V:

$$N(P) = V^\perp = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

3) Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$: $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.
We want to find $\mathcal{M}_{\mathcal{E}}^{\mathcal{V}} = \begin{bmatrix} T(\vec{e}_1)|_{\mathcal{V}} & T(\vec{e}_2)|_{\mathcal{V}} & T(\vec{e}_3)|_{\mathcal{V}} \end{bmatrix}.$

Using 2),

$$T(\vec{e}_1)|_{\mathcal{E}} = P\vec{e}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \quad T(\vec{e}_2)|_{\mathcal{E}} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad T(\vec{e}_3)|_{\mathcal{E}} = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

Finally, we need to find the coordinates of these vectors in $\mathcal{V}$:

$$M_{\mathcal{V}} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M_{\mathcal{V}}\, T(\vec{e}_1)|_{\mathcal{V}} = T(\vec{e}_1)|_{\mathcal{E}},$$

$$M_{\mathcal{V}}\, T(\vec{e}_2)|_{\mathcal{V}} = T(\vec{e}_2)|_{\mathcal{E}},$$

$$M_{\mathcal{V}}\, T(\vec{e}_3)|_{\mathcal{V}} = T(\vec{e}_3)|_{\mathcal{E}}.$$

We solve the three systems at once using Gauss-Jordan:

$$\left[\begin{array}{cc|ccc} 1 & 1 & 2/3 & -1/3 & -1/3 \\ -1 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & -1 & -1/3 & -1/3 & 2/3 \end{array}\right] \to \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1/3 & -2/3 & 1/3 \\ 0 & 1 & 1/3 & 1/3 & -2/3 \end{array}\right] \Rightarrow$$

$$\Rightarrow T(\vec{e}_1)|_{\mathcal{V}} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}, \quad T(\vec{e}_2)|_{\mathcal{V}} = \begin{bmatrix} -2/3 \\ 1/3 \end{bmatrix}, \quad T(\vec{e}_3)|_{\mathcal{V}} = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}.$$

$$M_{\mathcal{E}}^{\mathcal{V}} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \frac{1}{3}.$$

<u>Check:</u> $M_{\mathcal{E}}^{\mathcal{V}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} //$. $M_{\mathcal{E}}^{\mathcal{V}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.