MATH 312 LECTURE 4 : Vector Spaces and Subspaces.

We ended last lecture defining what a vector space and subspace is.

The most natural way to check in practice if a set V is a vector space is to show two properties:

1) If $\vec{v}, \vec{w} \in V$, then $\vec{v} + \vec{w} \in V$ (closed under addition),

2) If $\vec{v} \in V$, then $c\vec{v} \in V$ for any scalar c (closed under scaling).

(<u>both</u> need to be satisfied)

<u>Example:</u>

1) $\mathbb{R}^n$ is a vector space (yes, ok...)

2) Is $V = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$ a vector subspace of $\mathbb{R}^2$?

We can easily prove that it isn't (because it is not itself a vector space). Indeed,

$(1,1) \in V$, but $2(1,1) = (2,2) \notin V$ (since $2 \neq 2^2$).
$(1 = 1^2)$

(so we have shown it is not closed under scaling, therefore not a vector space).

3) Is $V = \{(x,y) \in \mathbb{R}^2 : y = 2x\}$ a vector space?

We can check for example that

$$\left.\begin{array}{l}(1,2) \in V \\ (2,4) \in V\end{array}\right\} \Rightarrow (1,2) + (2,4) = (3,6) \in V \text{ (as } 6 = 2\cdot 3)$$

and $3(1,2) = (3,6) \in V$,

but this is not proving that V is a vector space.

Properties 1) and 2) (addition and scaling) must hold for all vectors in the set and all scalars.

How to prove that V is a vector space?

Let $(x_1, y_1) \in V$, $(x_2, y_2) \in V$ (two points in V, that is, such that

$$y_1 = 2x_1, \quad y_2 = 2x_2.$$

We have to show that:

1) $(x_1, y_1) + (x_2, y_2) \in V \rightsquigarrow (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$y_1 + y_2 = 2(x_1 + x_2)? \text{ Yes } \checkmark$$

2) $\left.\begin{array}{l}(x_1, y_1) \in V \rightsquigarrow y_1 = 2x_1 \\ \text{Take } c \text{ any scalar}\end{array}\right\} \Rightarrow c(x_1, y_1) = (cx_1, cy_1) \in V$? Yes: $cy_1 = 2(cx_1)$ //.

- Notice that in the second example the set V $(y = 2x)$ was a line going through the origin.

  In general, the only vector subspaces of $\mathbb{R}^2$ are:

(1) The zero set: $\{(x,y) \in \mathbb{R}^2 : x = 0 = y\} \equiv \{\vec{0}\}$

(2) Lines through the origin.

(3) $\mathbb{R}^2$ itself.

<u>Example:</u>

- Is $V = \{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 x_2 = 0\}$ a subspace of $\mathbb{R}^2$?

  No: Counterexample

  $$\left.\begin{aligned} &\hookrightarrow \ \vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V \\ &\quad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V \end{aligned}\right\} \text{ But } \vec{v} + \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V, \text{ since } (v_1 + u_1)(v_2 + u_2) = 1 \cdot 1 \neq 0.$$

- Let's now do some more abstract examples.

<u>Example:</u> Is the set of functions $f : \mathbb{R} \to \mathbb{R}$ that are 2π periodic (i.e., $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$) a vector space?

Yes. Proof: Denote such set as S.

1) Let $f, g \in S$ (that is, $f(x+2\pi) = f(x)$, $g(x+2\pi) = g(x)$ for all $x \in \mathbb{R}$).

Is $f+g \in S$?

We have to check if:

$$(f+g)(x+2\pi) = f(x+2\pi) + g(x+2\pi) \underset{\substack{\uparrow \\ f,g \in V}}{=} f(x) + g(x) = (f+g)(x) \checkmark.$$

So it is closed under addition.

2) Let $f \in S$, $c \in \mathbb{R}$ (scalar) $\rightarrow$ Is $cf \in S$?

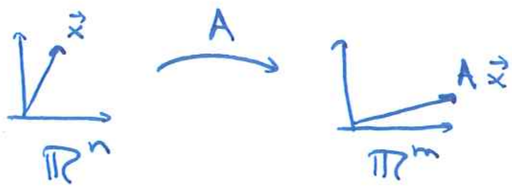
$$(cf)(x+2\pi) = c \cdot f(x+2\pi) \underset{\substack{\uparrow \\ f \in V}}{=} c f(x) = (cf)(x) \checkmark$$ closed under scaling. //.

- For home: Is the set of 2x2 symmetric matrices a vector space?

• We are going to introduce two important subspaces associated to a matrix: the column space and the nullspace.

I.1 The Column Space, $C(A)$

Consider a matrix A with m rows and n columns. That is,



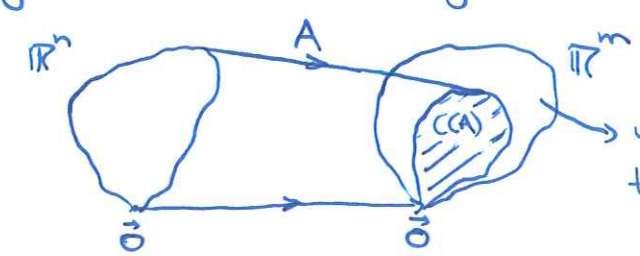
$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix}}_{\substack{A \\ (2\times 3)}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\substack{\vec{x} \\ \mathbb{R}^3}} = \underbrace{\begin{bmatrix} x_1 + x_3 \\ 3x_1 - x_2 + x_3 \end{bmatrix}}_{A\vec{x} \quad \mathbb{R}^2}$$

Def: Column Space

The column space of an $m\times n$ matrix is the collection of all vectors in $\mathbb{R}^m$ that can be output that matrix. That is,

$$C(A) = \{ \vec{b} \in \mathbb{R}^m : \exists \vec{x} \in \mathbb{R}^n \text{ with } A\vec{x} = \vec{b} \}$$

→ In other words, it is the "image" of the linear function described by A.



|| All $\mathbb{R}^n$ is sent to $C(A)$ by A.

→ vectors here can never be reached through A.

Remark: $C(A)$ is always a subspace of $\mathbb{R}^m$.

(Exercise for home: Prove that $C(A)$ is a subspace (of $\mathbb{R}^m$)).

- Recall the column picture:

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & | & \cdots & | & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n}_{\text{linear combination of the columns of } A.}$$

So we have

- Def. 2: The column space of A consists of all the linear combinations of the columns of A.

Remark: Important consequence

$$\| \; A\vec{x} = \vec{b} \text{ has a solution} \Leftrightarrow \vec{b} \in C(A) \; \|$$

- So, in practice, how can we describe the column space of a given matrix A?

Ex: $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$, is $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ in $C(A)$?

<u>Exercise:</u> Find the equations that any vector $\vec{b}$ has to verify to be in $C(A)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}$$

<u>Sol:</u>

$$\vec{b} \in C(A) \iff A\vec{x} = \vec{b} \text{ has a solution.}$$

When does $A\vec{x} = \vec{b}$ has a solution?

Do elimination:

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 0 & b_2 \\ 2 & 1 & b_3 \end{array}\right] \xrightarrow[R3' = R3 - 2R2]{R2' = R2 - 3R1} \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -6 & b_2 - 3b_1 \\ 0 & -3 & b_3 - 2b_1 \end{array}\right] \xrightarrow{R3'' = R3' - \frac{1}{2}R2'}$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -6 & b_2 - 3b_1 \\ 0 & 0 & b_3 - 2b_1 - \frac{1}{2}(b_2 - 3b_1) \end{array}\right]$$

So there is a solution if $b_3 - 2b_1 - \frac{1}{2}b_2 + \frac{3}{2}b_1 = 0$.

In summary,

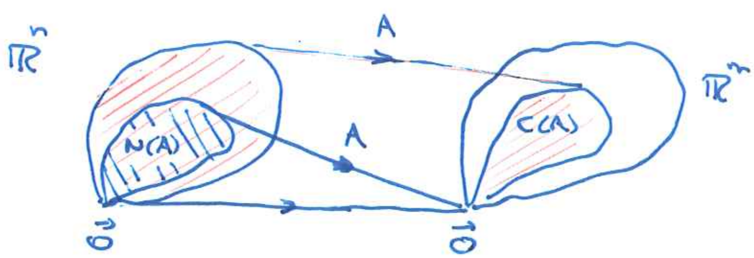
$$C(A) = \left\{ \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : b_3 - \frac{1}{2}b_2 - \frac{1}{2}b_1 = 0 \right\}.$$

In other words, $C(A)$ here is a plane (one equation in $\mathbb{R}^3$)

II.) The Nullspace, $N(A)$

Def: The nullspace of an $m\times n$ matrix A is the collection of all vectors in $\mathbb{R}^n$ that are sent ~~by~~ to the zero vector in $\mathbb{R}^m$ by A, i.e.,

$$N(A) = \{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \}$$



A "Kills" $N(A)$.

- Q: How to find $N(A)$?

Well, we know that: Solve $A\vec{x} = \vec{0}$ ||

Example: Let

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 1 & 0 & 5 \\ 3 & 4 & 5 & 5 \\ 1 & -2 & -5 & 5 \end{bmatrix}.$$

1) Is $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in $N(A)$?

2) Find all vectors in $N(A)$

1) $\vec{u} \in N(A)$ if $A\vec{u} = \vec{0}$. Let's check it

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ & \ddots & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ \vdots \\ \\ \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{u} \notin N(A)$$

2) We want to find all $\vec{x}$ / $A\vec{x} = \vec{0}$: (solve $A\vec{x} = \vec{0}$)

→ Perform Gauss-Jordan until we obtain the RREF.

→ "Read" the solution introducing parameters.

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 2 & 1 & 0 & 5 \\ 3 & 4 & 5 & 5 \\ 1 & -2 & -5 & 5 \end{bmatrix} \xrightarrow[\substack{R3' = R3 - 3R1 \\ R4' = R4 - R1}]{R2' = R2 - 2R1} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & -5 & -10 & 5 \\ 0 & -5 & -10 & 5 \\ 0 & -5 & -10 & 5 \end{bmatrix} \xrightarrow[R4'' = R4' - R2']{R3'' = R3' - R2'} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & -5 & -10 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{R2'' = \frac{R2'}{-5}} \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1' = R1 - 3R2''} \left.\begin{bmatrix} \boxed{1} & 0 & -1 & 3 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \text{RREF}$$

free variables

So,

$$\left.\begin{aligned} x_3 &= \alpha \\ x_4 &= \beta \\ x_2 &= -2\alpha + \beta \\ x_1 &= \alpha - 3\beta \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Big\} \text{ This describes all possible } \vec{x} \in N(A).$$

we will see that these are a "basis" of $N(A)$.

<u>Exercise:</u> Construct a matrix with

$$\begin{bmatrix}1\\2\\1\end{bmatrix} \in C(A) \quad \text{and} \quad \begin{bmatrix}1\\1\end{bmatrix} \in N(A).$$

Solution:

- First, n° of cols and rows of A?

Recall that $A\ m\times n \to \left.\begin{matrix} C(A) \subseteq \mathbb{R}^m \\ N(A) \subseteq \mathbb{R}^n \end{matrix}\right\}$ so here $m=3$, $n=2$.

Now, we want $\begin{bmatrix}1\\2\\1\end{bmatrix}$ be a column of A:

$A = \begin{bmatrix}1 & a\\2 & b\\1 & c\end{bmatrix}$, and need to find some a, b, c such that

$$\begin{bmatrix}1 & a\\2 & b\\1 & c\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \left.\begin{matrix} 1+a=0 \\ 2+b=0 \\ 1+c=0 \end{matrix}\right\} \Rightarrow \begin{matrix} a=-1 \\ b=-2 \\ c=-1 \end{matrix}$$

So $A = \begin{bmatrix}1 & -1\\2 & -2\\1 & -1\end{bmatrix}$ (there are other possible choices).

• The complete solution to $A\vec{x} = \vec{b}$:

If we know one particular solution $\vec{x}_p$ ($A\vec{x}_p = \vec{b}$), then all the solutions are given by

$\vec{x} = \vec{x}_p + \vec{x}_n$, where $\vec{x}_n$ is the solution to the homogeneous problem $A\vec{x} = \vec{0}$.

that is "the nullspace of A"

$$\left(\text{Why? } A(\vec{x}_p + \vec{x}_n) = \underbrace{A\vec{x}_p}_{\vec{b}} + \underbrace{A\vec{x}_n}_{\vec{0}} = \vec{b}\right)$$

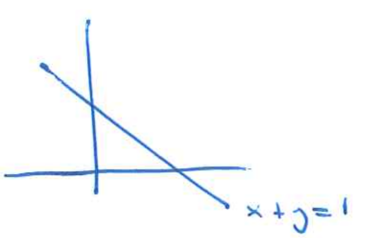
Therefore, we have that for $A\vec{x} = \vec{b}$

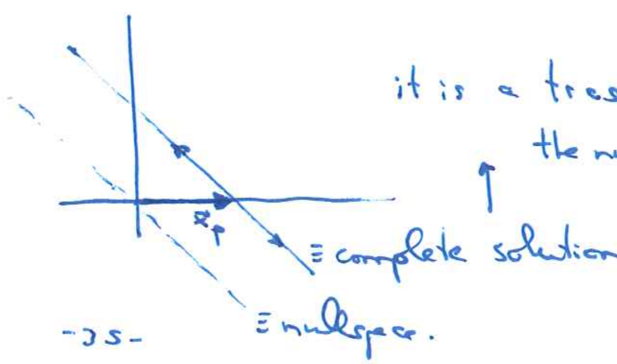
1) $\exists$ solution if $\vec{b} \in C(A)$.

2) If $N(A) = \{\vec{0}\}$ (no free variables), ~~the~~ at most one solution.

We can visualize the complete solution:

Ex:
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{x}_p} + a\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{nullspace}}.$$





it is a translation of the nullspace.

• <u>Linear indepence</u>

<u>Def</u>: A set of vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ is said to be linearly independent if

$$\vec{0} = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_n\vec{u}_n \Rightarrow c_1 = c_2 = \dots = c_n = 0.$$

Otherwise, they are called linearly dependent.

<u>Examples</u>:

1) $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (ok, we know)

Using definition:

$$c_1\vec{u}_1 + c_2\vec{u}_2 = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} c_1 = 0 \\ c_2 = 0 \end{matrix}\right\} \text{ only possible solution} \Rightarrow \text{l.i.}$$

2) $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \begin{bmatrix} c_1 + 2c_2 + 3c_3 \\ 0 \\ c_1 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} c_1 + 2c_2 + 3c_3 = 0 \\ c_1 + c_3 = 0 \end{cases}$$

This final system has infinite solutions, so there are many choices of c_1, c_2, c_3 (not just $c_1 = c_2 = c_3 = 0$) $\Rightarrow$ l. dependent.

3) [General Method] How to decide if a given set of vectors are l.i. or l.d.?

<u>Answer:</u>

$\vec{u}_1, \dots, \vec{u}_n$ are l.i. $\Leftrightarrow$

$\{ c_1\vec{u}_1 + \dots + c_n\vec{u}_n = \vec{0} \Rightarrow c_1 = \dots = c_n = 0 \} \Leftrightarrow$

$\Leftrightarrow \left\{ \left[\vec{u}_1 | \dots | \vec{u}_n \right] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\} \Leftrightarrow$

$\Leftrightarrow \left[\vec{u}_1 | \dots | \vec{u}_n \right]$ has zero nullspace $\Leftrightarrow$

$\Leftrightarrow$ the RREF of $\left[\vec{u}_1 | \dots | \vec{u}_n \right]$ has no free variables, i.e., all columns have pivots.

So, in summary,

Given $\vec{u}_1, \dots, \vec{u}_n$, to decide if they are l.i.:

1) Create the matrix $\left[\vec{u}_1 | \dots | \vec{u}_n \right]$

2) Elimination to get RREF.

3) Every column has a pivot? $\nearrow$ Yes: vectors are l.i.
$\searrow$ No: vectors are l.d.