MATH 312
LECTURE 5

: Spanning sets and Basis.

- Last day: 1) Checking if a set V is a vector space.
  2) Two important subspaces associated to a matrix A: - Column Space $C(A)$ - Nullspace $N(A)$.
  3) Complete Solution to $A\vec{x} = \vec{b}$.

- The Complete solution to $A\vec{x} = \vec{b}$

<u>Example</u>: Find the complete solution to

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{b}}$$

<u>Sol</u>: $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R2' = R2 - 2R1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 1 & 1 \\ 0 & \boxed{1} & -1 & 0 & -1 \end{array}\right]$ <u>RREF</u>

pivots   free variables

Thus,

$$\begin{matrix} x_3 = \alpha \\ x_4 = \beta \end{matrix} \rightarrow \begin{matrix} x_1 = 1 - \alpha - \beta \\ x_2 = -1 + \alpha \end{matrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \underbrace{\alpha \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{nullspace of } A}.$$

• Possible situations:

<u>Def</u>: Rank of A

The rank of a matrix A is the number of pivots.

Cases on solutions: (Let A $m \times n$) Two important cases:

1) Full column rank: $r = n$ ($n^{\underline{o}}$ of pivots $= n^{\underline{o}}$ of columns)

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \leadsto \underbrace{\begin{bmatrix} \square & & \\ & \square & \\ & & \square \\ 0 \cdots & 0 & 0 \\ & \vdots & \end{bmatrix}}_{\text{RREF of } A}$$

All columns have pivots $\Leftrightarrow$

$\Leftrightarrow$ No free variables $\Leftrightarrow N(A) = \{\vec{0}\}$

$\Rightarrow$ <u>At most</u> one solution.

*) There might not be any solution: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$!

2) Full row rank: $r = m$ ($n^{\underline{o}}$ of pivots $= n^{\underline{o}}$ of rows)

$$\begin{bmatrix} \square & & & \cdots \\ & \square & & \cdots \\ & & \square & \cdots \end{bmatrix}$$

All rows have pivots $\Rightarrow A\vec{x} = \vec{b}$ always have a solution $\Rightarrow$

$\Rightarrow C(A) = \mathbb{R}^m$

$\uparrow$ possible free variables (in particular, $n - m$ free variables).

• In summary, 4 cases depending on rank, n, m:

1) $r = n = m$ (full rank)

$$\begin{bmatrix} \square & & \\ & \square & \\ & & \square \end{bmatrix} \left.\begin{matrix}\text{A is square} \\ \text{and invertible}\end{matrix}\right\} \begin{matrix} A\vec{x} = \vec{b} \text{ has one and only one} \\ \text{solution.}\end{matrix}$$

2) $r = n,\ r < m$

$$\begin{bmatrix} \square & & \\ & \square & \\ & & \square \\ \text{zero} & \text{rows} & \end{bmatrix} \Bigg\} \ A\vec{x} = \vec{b} \text{ has 0 or 1 solution.}$$

3) $r = m,\ r < n$

$$\begin{bmatrix} \square & & & \cdots \\ & \square & & \cdots \\ & & \square & \underbrace{\cdots}_{\substack{\text{free} \\ \text{variables}}} \end{bmatrix} \Bigg\} \ A\vec{x} = \vec{b} \text{ has } \infty \text{ solutions.}$$

($N(A)$ contains nonzero vectors).

4) $r < n,\ r < m$

$$\begin{bmatrix} \square & & \cdots \\ & \square & \cdots \\ \text{zero} & \text{rows} & \end{bmatrix} \Bigg\} \ A\vec{x} = \vec{b} \text{ has 0 or } \infty \text{ solutions.}$$

$\uparrow$
free variables

• Linear independence.

Example: (No column vectors)

1) Are the matrices A, B linearly independent?

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol: They are l.i. if

$$\underset{\substack{\uparrow \\ \text{zero matrix } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}}{c_1 A + c_2 B = 0} \Rightarrow \underset{\substack{\uparrow \\ \text{number}}}{c_1 = c_2 = 0}.$$

Let's check it:

$$c_1 A + c_2 B = \begin{bmatrix} c_1 & 2c_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 & 2c_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 & 2c_1 + 2c_2 \\ 0 & c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} c_1 + 2c_2 &= 0 \\ 2c_1 + 2c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}\right\} \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\underline{c_1 = c_2 = 0}}.$$

So yes, A, B are l.i.

2) Are the polynomials $x, x^2+2, 3x^2, 4x+5x^2$ lin. dep.?

They are lin. dep. if they are not lin. indep.

They are lin. indep. if

$$c_1 x + c_2(x^2+2) + c_3(3x^2) + c_4(4x+5x^2) = \underset{\text{zero polynomial (i.e., 0 for all } x)}{0} \Rightarrow c_1 = c_2 = c_3 = c_4 = 0.$$

Let's check it:

$$= x(c_1 + 4c_4) + x^2(c_2 + 3c_3 + 5c_4) + (2c_2) = \overset{\forall x}{0} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c_1 + 4c_4 = 0 \\ c_2 + 3c_3 + 5c_4 = 0 \\ 2c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 + 4c_4 = 0 \\ 3c_3 + 5c_4 = 0 \\ c_2 = 0 \end{cases} \Rightarrow \text{Infinite solutions!}$$

So the polynomials were lin. dep.

<u>Def</u>: Spanning set

Let V be a vector space and $\vec{u}_1, \dots, \vec{u}_k$ a collection of vectors in V.

We say that $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a spanning set of V (or that $\vec{u}_1, \dots, \vec{u}_k$ spans V) if every vector $\vec{b} \in V$ is a linear combination of $\vec{u}_1, \dots, \vec{u}_k$.

That is,

$\vec{u}_1, \dots, \vec{u}_k$ spans $V \Leftrightarrow \forall \vec{b} \in V, \exists c_1, \dots, c_k$ scalars such that

$$\vec{b} = c_1\vec{u}_1 + \dots + c_k\vec{u}_k$$

• Examples:

1) $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ don't span $\mathbb{R}^2$.

Indeed, for $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we cannot find $c_1, c_2 \in \mathbb{R}$ such that $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, since $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ doesn't have a solution.

2) $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^3$.

To show it, we need to check that for any $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$, we can find $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$\vec{b} = c_1 \vec{u}_1 + \dots + c_4 \vec{u}_4$.

That is, we need to check that the system

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ \vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3 \quad \vec{u}_4 \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution no matter which $\vec{b}$.

→ This will be true if every row has a pivot:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}.$$

3) Does $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ spans the vector space

$$V = \left\{ \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3 : b_1 + b_2 + b_3 = 0 \right\} \quad ?$$

Sol:

Need to check that any $\vec{b}$ with $b_1 + b_2 + b_3 = 0$ can be written as

$\vec{b} = c_1 \vec{u}_1 + c_2 \vec{u}_2$ for some $c_1, c_2 \in \mathbb{R}$.

That is, that the system

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 - b_2 \end{bmatrix}$$ always has a solution.

Elimination gives:

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ -1 & 1 & b_2 \\ 0 & -1 & -b_1-b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2+b_1 \\ 0 & -1 & -b_1-b_2 \end{array}\right] \to \left[\begin{array}{cc|c} \boxed{1} & 0 & b_1 \\ 0 & \boxed{1} & b_2+b_1 \\ 0 & 0 & 0 \end{array}\right]$$

So yes, always a solution $\rightarrow$ $\vec{u}_1, \vec{u}_2$ spans V.

• Def: Basis

Let V be a vector space. The vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ are a basis of V if

1) They are linearly independent ($\sim$ i.e. "not too many")

2) They span V (i.e. "not too few").

Remark: In $\mathbb{R}^n$, all basis contain $\underline{n}$ vectors.

- If there are less, they cannot span $\mathbb{R}^n$
- If there are more, they cannot be l.i.

• Exercise: Do the vectors $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

form a basis of $\mathbb{R}^3$?

Solution: Are they lin. indep. and span $\mathbb{R}^3$?

1) Lin. indep. if $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ only has $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as a solution.

2) Span $\mathbb{R}^3$ if $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ always has a solution (no matter which $\vec{b}$).

Notice that

1) Is equivalent to say that $N(A) = \{\vec{0}\}$, or in other words, no free variables $\to$ That is, all columns must have a pivot.

2) Is equivalent to say that all rows have a pivot.

So we are asking if $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has full rank:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{-2} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix} \quad \text{yes}$$

• Exercise: Find a basis for the subspace

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = x \right\}$$

Solution:

We are looking for all the vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that satisfy $x = y$, that is,

$$x - y = 0 \longrightarrow \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \text{. Denote } A = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

So we are looking for $N(A)$!

$$\begin{bmatrix} \boxed{1} & \underset{\substack{\uparrow \\ \text{free variable}}}{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \rightarrow \begin{matrix} y = \alpha \\ x = \alpha \end{matrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Our basis is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ (obviously lin. indep., and it spans V)

Exam-like quizz: True or false (Explan why) (9) — 436 responses

1. If a 3×4 matrix has RREF with only three pivots, then its rows are linearly dependent. — 51 responses
2. If a 3×4 matrix has RREF with three pivots, then its columns must span $\mathbb{R}^3$. — 51 responses
3. If a matrix A satiesfies $Ax = 0$ for some $x \neq 0$, then A cannot be invertible. — 49 responses
4. The set consisting of the X axis, the Y axis and the line $y = x$ is a subspace of $\mathbb{R}^2$. — 48 responses
5. If a vector k lies in the nullspace of A^T and if $Ax = b$, then $A(x + k) = b$. — 48 responses
6. If E is the elementary matrix which adds 3 times row 1 to row 2, then E^2 add 9 times row 1 to row 2. — 47 responses

→x 7. ~~The matrices A and $-A$ have the same four subspaces.~~ — 49 responses

→x 8. ~~The matrices A and A^T always have the same number of pivots.~~ — 47 responses

9. The polynomials $1 + x$, $2x^2$ and $2 + 2x + x^2$ form a basis for the subspace of polynomials with degree no mo... — 46 responses