MATH 312 LECTURE 6	: Four Fundamental Subspaces of A

Let A be an $m \times n$ matrix. There are "Four Fundamental Subspaces" associated to it:

1) $C(A)$ in $\mathbb{R}^m$ ≡ Column Space ≡ Span of the columns of A.

2) $N(A)$ in $\mathbb{R}^n$ ≡ Nullspace ≡ Vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$.

3) $C(A^T)$ in $\mathbb{R}^n$ ≡ Row Space ≡ Span of the rows of A

4) $N(A^T)$ in $\mathbb{R}^m$ ≡ Left Nullspace ≡ Vectors $\vec{x}$ such that $\vec{x}^T A = \vec{0}^T$.

• <u>Goal</u>: 1) To find basis for $C(A)$, $C(A^T)$, $N(A)$, $N(A^T)$

2) To relate the dimensions of the four subspaces.

Recall that

<u>Def</u>: Dimension of a vector space V.

$\dim V$ = nº of elements in a basis of V.

$\boxed{\text{I}}$ We will start with matrices R that are in RREF.

For example,

$$R = \begin{bmatrix} \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \left(\begin{matrix} m = 4 \\ n = 5 \end{matrix}, \ \text{rank}(R) = r = 3 \right)$$

1) $\underline{C(R)}$

Note that:

- → Columns with pivots are l.i.
- → Non-pivot columns can be written as lin. comb. of pivot ones.
  - ↳ Pivot columns span $C(R)$.

Therefore,

A basis of $C(R)$ = "pivot columns of R" $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim C(R) = 3$ (we knew this, since $r = 3$).

2) $\underline{N(R)}$

It is given by the free variables when solving $R\vec{x} = \vec{0}$.

That is,

$$\begin{matrix} x_2 = \alpha \\ x_4 = \beta \end{matrix} \longrightarrow \begin{matrix} x_1 = -2\alpha - \beta \\ x_3 = -2\beta \\ x_4 = 0 \end{matrix} \longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

→ these vectors have to be l.i.

they span $N(R)$

So basis of $N(R) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$, and $\dim N(R) = 2$.

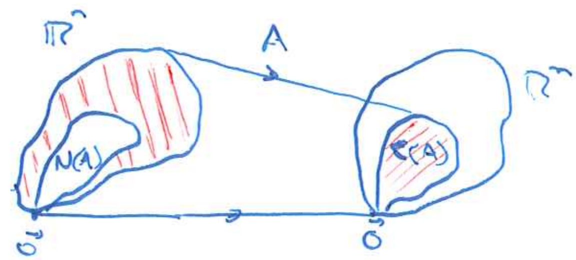
Note that $\dim N(R) = n - r = 5 - 3 = 2$.

- <u>Rank-Nullity Theorem</u>: Let A be any $m \times n$ matrix with $\operatorname{rank}(A) = r$. Then,

$$\| \; n = r + \dim N(A) \; \|$$

This can also be written as:

$$\underbrace{\dim(\text{domain}(A))}_{n} = \underbrace{\dim N(A)}_{n-r} + \underbrace{\dim C(A)}_{r}$$



$$\dim C(A) = n - \dim N(A)$$

3) <u>$C(R^T)$</u>

Note that the rows with pivots are l.i., and that the rest are zero rows. So the pivot rows form a basis:

Basis for $C(R^T) =$ "rows with pivots" $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Also, $\dim C(R^T) = n^{\circ}$ of pivots $= r = \dim C(R)$ $\|$.

4) $\underline{N(R^T)}$

Q: $\dim N(R^T)$?

By rank-nullity theorem applied to R^T, we find that:

$m = \dim C(R^T) + \dim N(R^T) \rightarrow \dim N(R^T) = m - \dim C(R^T) =$
$= m - r$

Here,

$\dim N(R^T) = 4 - 3 = 1$.

• $\|$ Summary of dimensions: Fundamental Theorem of Linear Algebra (part I)

For any matrix A $m \times n$ with $\mathrm{rank}(A) = r$, it holds that

$$\left.\begin{array}{l} \dim C(A) = r = \dim C(A^T) \\ \dim N(A) = n - r \\ \dim N(A^T) = m - r \end{array}\right\}$$

Basis for $N(R^T)$:

$$\vec{x}^T R = \vec{0} \rightarrow [x_1 \;\; x_2 \;\; x_3 \;\; x_4] \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [x_1 \;\; 2x_1 \;\; x_2 \;\; x_1 + 2x_2 \;\; x_3] = [0\,0\,0\,0\,0]$$

$\Downarrow$

$x_1 = 0 = x_2 = x_3$

$\leftarrow$ $\underline{\underline{x_4 \text{ free}}}$

$$\text{basis } N(R^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\boxed{2}$ We do it now for a general matrix A $(m \times n)$.

For example,

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We want to know how to obtain $C(A)$, $N(A)$, $C(A^T)$, $N(A^T)$.

We will see two ways:
- → directly
- → Using the RREF of A.

(i.e., $A \underset{\text{row operations}}{\rightsquigarrow} R$, i.e., $\overbrace{E}^{\text{invertible}} A = R$ with E = product of elementary matrices)

1) $\underline{N(A)}$

- This is obtained by solving $A\vec{x} = \vec{0}$. The columns given by the parameters will form a basis.
- Notice that if we have R (the RREF of A), then

$$N(A) = N(R)$$

This is clear because: $A\vec{x} = \vec{0} \Leftrightarrow EA\vec{x} = \vec{0} \Leftrightarrow R\vec{x} = \vec{0}$
$\qquad\qquad\qquad\qquad\uparrow$ (E is invertible)

Also, think that from $A\vec{x} = \vec{0}$ to $R\vec{x} = \vec{0}$ we are just doing elementary operations on the equations (the solution doesn't change).

2) $C(A)$

Is $C(A) = C(R)$? NO

We can see it in our example:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(RREF)} R = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the columns of R can never produce this column of A!

- However, the position of the linearly independent columns of A is the same as the position of the pivot columns in R.

Why?

$$A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \rightsquigarrow R = \begin{bmatrix} E\vec{a}_1 & \cdots & E\vec{a}_n \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad \underset{\vec{r}_1}{\downarrow} \quad \cdots \quad \underset{\vec{r}_n}{\downarrow} \text{ (columns of R)}$$

Now, E is invertible, so $E\vec{x} \neq \vec{0}$ for all $\vec{x} \neq \vec{0}$.

One can check (exercise) that for E invertible,

$\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ are l.i. $\Leftrightarrow$ $\{E\vec{u}_1, E\vec{u}_2, \ldots, E\vec{u}_k\}$ are l.i.

In summary,

$\dim C(A) = r = \dim C(R)$

Basis for $C(A)$ = "columns <u>of A</u> in the position of the pivots in R".

<u>Remark</u>: To find a basis of $C(A)$ one needs to do elimination to find the position of the pivot columns.

In our example,

$$\text{Basis of } C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

3) <u>$C(A^T)$</u>

This is not changed by row operations. Indeed, note that $EA = (\underbrace{E_1 \dots E_n}_{\text{row operations}})A = R$ is only doing linear combinations of the rows of A. So span{rows of A} = = span{rows of R}.

In conclusion,

$$C(A^T) = C(R^T)$$ (i.e., one needs to do elimination to find $C(A^T)$).

4) $\underline{N(A^T)}$

By definition, $\vec{x} \in N(A^T)$ if $\vec{x}^T A = \vec{0}^T$. This is the same as $A^T\vec{x} = \vec{0}$.

That is, the left nullspace is simply the nullspace of A^T.

↳ So, one way to find a basis for $N(A^T)$ is finding the nullspace of A^T (perform Gauss-Jordan, free variables, ...).

- But, if we are given (or if we have computed them in advance) the matrices E, R such that

$EA = R$,

then we don't need to compute anything more!

Why?

First, recall that $\dim N(A^T) = m-r$, so we only need $m-r$ vectors to find a basis of $N(A^T)$.

Notice that R has r pivots and m rows, so the last $m-r$ rows are all zeros:

$$EA = \underbrace{\begin{bmatrix} \vec{e}_1^T \\ \hline \vdots \\ \hline \vec{e}_m^T \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \vec{e}_1^T A \\ \hline \vdots \\ \hline \vec{e}_m^T A \end{bmatrix}}_{EA} = \underbrace{\begin{bmatrix} \\ \cdots\cdots \\ \text{zero rows} \end{bmatrix}}_{R} \Big\} m-r$$

So just by looking at the $m-r$ rows we see that

$\vec{e}_j^{\,T} A = \vec{0}^{\,T} \quad (j = m, m-1, \dots)$

That is:

- <u>Summary</u>: The last $m-r$ rows of E (in $EA=R$) form a basis of the left nullspace of A, $N(A^T)$.

<u>Remark</u>: If we don't know E, we might prefer to simply solve $A^T \vec{x} = \vec{0}$.

↳ Notice that if we have $A = LU$, we can do the same reasoning with L^{-1}.

- Coordinates and change of basis

We know that in $\mathbb{R}^n$ all vector subspaces are lines, planes,... going through the origin.

How does vector subspaces of polynomials look like?

Example: Let $\mathbb{P}_2 = \{p \text{ polynomial} : p(x) = a + bx + cx^2\}$ $(a,b,c \in \mathbb{R})$.

and let $V = \{p \text{ polynomial} : p(x) = a + bx\}$,

$V' = \{p \text{ pol.} : p(x) = a + ax\}$

We can check that both V, V' are subspaces of $\mathbb{P}_2$.

For example, for V':

1) Addition: Let $p_1, p_2 \in V'$. That is, $p_1(x) = a_1 + a_1 x$, $p_2(x) = a_2 + a_2 x$. Then, $(p_1 + p_2)(x) = p_1(x) + p_2(x) =$

$= a_1 + a_1 x + a_2 + a_2 x =$

$= (a_1 + a_2) + (a_1 + a_2)x \Rightarrow p_1 + p_2 \in V'$.

2) Scaling: $p_1 \in V'$ $(p_1(x) = a_1 + a_1 x)$, $c \in \mathbb{R}$. Then $(cp_1)(x) = c\,p_1(x) =$

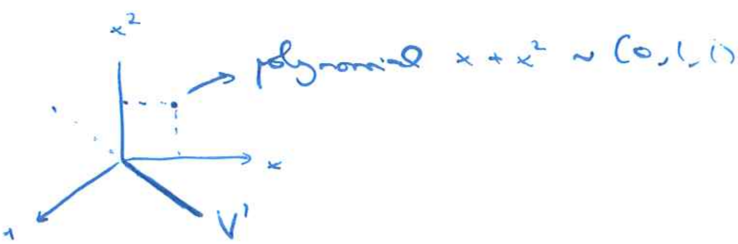
$= c(a_1 + a_1 x) = (ca_1) + (ca_1)x \Rightarrow cp_1 \in V'$.

Now, we want to draw polynomials of $\mathbb{P}_2$.

We are going to identify

$$p(x) = a + bx + cx^2 \quad \text{with} \quad \vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This way, each polynomial in $\mathbb{P}^2$ can be represented as a point in $\mathbb{R}^3$:



What is V'? Polynomials like $a + ax \rightsquigarrow$ points like $(a,a,0) =$

$= a(1,1,0)$

$\Vert$ This is a line

Similarly, we can think of V as the "horizontal" plane:

$$p \in V \to p(x) = a + bx \rightsquigarrow \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Basically, we are picking the basis $\{1, x, x^2\}$ for $\mathbb{P}_2$, and writing all the vectors (polynomials) using their "coordinates" in this basis.

Def: Coordinates

Let $B = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be an (ordered) basis of a vector space V.

Any vector $\vec{v} \in V$ is written (in a unique way) as a lin. combination of these vectors:

$\vec{v} = c_1 \vec{u}_1 + \dots + c_n \vec{u}_n$.

The scalars $c_1, \dots, c_n$ are the <u>coordinates</u> of $\vec{v}$ in B.

Moreover, we identify the vector $\vec{v}$ with its "coordinate vector":

$$\vec{v}\Big|_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

<u>Example</u>: Find the coordinates of $\vec{v} = \begin{bmatrix} 5 \\ 18 \end{bmatrix}$ in the basis

$$B = \{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}.$$

<u>Sol</u>:

$$\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 \to \begin{bmatrix} 5 \\ 18 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \to \text{solve the system.}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 5 & 3 & 18 \end{array}\right] \xrightarrow{R2' = R2 - \frac{5}{2} R1} \cancel{\left[\begin{array}{cc|c} 2 & -1 & 5 \\ 0 & 11/2 & \end{array}\right]} \dots \quad \begin{matrix} c_1 = 3 \\ c_2 = 1 \end{matrix} \to \vec{v}\Big|_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\cancel{3 + \frac{5}{2} = 1}$

Example: Find the coordinates of $p(x) = 5 + 5x - 2x^2$ in the basis $B = \{1, 1-x, 2-4x+x^2\}$.

Sol: Using the standard basis,

$$p \rightsquigarrow \vec{v} = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix}, \quad B = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

And we need to find c_1, c_2, c_3 such that

$$\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 \Rightarrow \text{Solve } \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- That is, in general:

1) Write the vectors $\vec{u}_1, \dots, \vec{u}_n$ in the standard basis, and create the matrix $M_B = \left[\vec{u}_1|_{\varepsilon} \middle| \dots \middle| \vec{u}_n|_{\varepsilon} \right]$.

2) Write $\vec{v}$ in the standard basis, $\vec{v}|_{\varepsilon}$.

3) Solve $\left\| M_B \, \vec{v}|_B = \vec{v}|_{\varepsilon} \right\|$

Remark: We won't generally write $\vec{v}|_{\varepsilon}$ when it is the standard basis.

- Change of basis (change of coordinates)

Notice that we already know how to change coordinates from the standard basis to a different one B.

- Now, given basis $B=\{\vec{b}_1,\dots,\vec{b}_n\}$ and $C=\{\vec{c}_1,\dots,\vec{c}_n\}$, we want to find a matrix that changes from the coordinates in B to those in C:

$$\vec{v}|_C = \underset{C\leftarrow B}{P}\,\vec{v}|_B .$$

Well, we can go through the standard basis:

$$\vec{v}|_B = M_B^{-1}\vec{v} \Rightarrow \vec{v} = M_B\,\vec{v}|_B$$

$$\vec{v}|_C = M_C^{-1}\vec{v} \longrightarrow \vec{v}|_C = M_C^{-1}M_B\,\vec{v}|_B \Rightarrow \left\| \underset{C\leftarrow B}{P} = M_C^{-1}M_B \right\|$$

- Direct way:

$$\left.\begin{aligned}\vec{v} &= \alpha_1\vec{b}_1+\dots+\alpha_n\vec{b}_n = \begin{bmatrix}\vec{b}_1 & \cdots & \vec{b}_n\end{bmatrix}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}\\ \vec{v} &= \gamma_1\vec{c}_1+\dots+\gamma_n\vec{c}_n = \begin{bmatrix}\vec{c}_1 & \cdots & \vec{c}_n\end{bmatrix}\begin{bmatrix}\gamma_1\\ \vdots\\ \gamma_n\end{bmatrix}\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow \begin{bmatrix}\vec{c}_1 & \cdots & \vec{c}_n\end{bmatrix}\begin{bmatrix}\gamma_1\\ \vdots\\ \gamma_n\end{bmatrix} = \begin{bmatrix}\vec{b}_1 & \cdots & \vec{b}_n\end{bmatrix}\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} \Rightarrow$$ Perform elimination until you obtain the identity on the left //