MATH 312
LECTURE 7

: Change of basis ~~and Linear Transformations~~.

<u>Exercise:</u> Consider the vector space of 2×2 matrices.

Write $A = \begin{bmatrix} 2 & -1 \\ -4 & -4 \end{bmatrix}$ in the basis $\mathcal{B}$,

$$\mathcal{B} = \Big\{ \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{B_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}_{B_2}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B_3}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{B_4} \Big\}$$

<u>Solution:</u>

[General method] As we did the other day, three steps:

1) Write the vectors in the basis $\mathcal{B}$ as column vectors (using a standard basis). Create a matrix with them.

<u>Remark:</u> Standard basis for space of 2×2 matrices?

$$\mathcal{E} = \Big\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Big\}$$

So,

$$B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

*) Here, we should be writing $B_1|_{\mathcal{E}}, B_2|_{\mathcal{E}}, \ldots$

Therefore, $M_B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

2) Write A in the standard basis: $(A|_{\mathcal{E}})$

$$A = \begin{bmatrix} 2 \\ -1 \\ -4 \\ -4 \end{bmatrix}$$

3) Solve $M_B \, A|_{\mathcal{B}} = A$:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -4 \\ -4 \end{bmatrix} \Rightarrow c_1, c_2, c_3, c_4 \,//.$$

- Remark: What were we really doing?

Apply the definition of coordinates of A in the basis $\mathcal{B}$:

$$A|_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \iff A = c_1 B_1 + c_2 B_2 + c_3 B_3 + c_4 B_4 .$$

Thus we can find $c_1, \dots, c_4$ from the identity

$$\begin{bmatrix} 2 & -1 \\ -4 & -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} c_1 + c_3 & c_1 + c_4 \\ c_2 + c_4 & c_2 + c_3 + c_4 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} 2 &= c_1 + c_3 \\ -1 &= c_1 + c_4 \\ -4 &= c_2 + c_4 \\ -4 &= c_2 + c_3 + c_4 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -4 \\ -4 \end{bmatrix} \Rightarrow \text{solve it} //.$$

- <u>Change of basis:</u>

Let V be a vector space. Say we have two basis of V, denoted B and C:

$B = \{\vec{b}_1, \dots, \vec{b}_n\}$, $C = \{\vec{c}_1, \dots, \vec{c}_n\}$.

We want to find a matrix $P_{C \leftarrow B}$ such that, given the coordinates of a vector in the basis B, we obtain the coordinates in C:

$\vec{v}|_C = P_{C \leftarrow B}\, \vec{v}|_B$.

→ How?

We already know how the coordinates in a basis relate to the ones in the standard basis, so

$$\left.\begin{array}{l}\vec{v} = M_C \, \vec{v}|_C \\ \vec{v} = M_B \, \vec{v}|_B\end{array}\right\} \Rightarrow \vec{v}|_C = M_C^{-1}\vec{v} = M_C^{-1} M_B \, \vec{v}|_B \quad !!$$

That is,

$$\boxed{P_{C \leftarrow B} = M_C^{-1} M_B}$$

<u>Example</u>: Let $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

Find the change of basis matrix $P_{C \leftarrow B}$.

<u>Solution</u>:

$$M_C = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \; M_B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow P_{C \leftarrow B} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} =$$

$$= - \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = - \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}.$$

- "Different way":

Recall what we are really doing,

If $B = \{\vec{b}_1, \vec{b}_2\}$, $C = \{\vec{c}_1, \vec{c}_2\}$ and we denote

$\vec{v}|_C = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, $\vec{v}|_B = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$, this means that:

$$\vec{v} = \alpha_1 \vec{c}_1 + \alpha_2 \vec{c}_2 = \left[\vec{c}_1 \,\vdots\, \vec{c}_2\right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

and

$$\vec{v} = \beta_1 \vec{b}_1 + \beta_2 \vec{b}_2 = \left[\vec{b}_1 \,\vdots\, \vec{b}_2\right] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}}_{M_C} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\vec{v}|_C} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{M_B} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\vec{v}|_B}$$

- We want to find $\vec{v}|_C \rightsquigarrow$ so do Gauss-Jordan until we have the identity:

$$\left[\underbrace{\begin{matrix} 1 & 0 \\ -1 & -1 \end{matrix}}_{M_C} \;\vdots\; \underbrace{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}}_{M_B}\right] \xrightarrow{\text{Gauss-Jordan}} \left[\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \;\vdots\; \begin{matrix} 1 & -1 \\ 2 & 0 \end{matrix}\right] \rightsquigarrow$$

$$\rightsquigarrow \left[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \;\vdots\; \underbrace{\begin{matrix} 1 & -1 \\ -2 & 0 \end{matrix}}_{M_C^{-1} M_B}\right]$$

Exercise: Let $B = \{1, 1+x, 1+x+x^2\}$, $C = \{1+x^2, x+x^2, x\}$

be two basis of $\mathbb{P}_2$ (vector space of polynomials with degree ≤ 2).

1) Find change of basis from B to C.

2) Use it to find $p|_C$, where $p(x) = $ ~~[illegible]~~ $3(1+x) + 1 + x + x^2$.

Sol:

1) $M_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $M_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$P_{C\leftarrow B} = M_C^{-1} M_B \quad \left(\text{Recall: } \left.\begin{matrix} M_B \vec{v}|_B = \vec{v} \\ M_C \vec{v}_C = \vec{v} \end{matrix}\right\} \Rightarrow \vec{v}|_C = M_C^{-1}\vec{v} = \underline{\underline{M_C^{-1} M_B}} \vec{v}|_B \right).$$

So $P_{C\leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

We do this in the "fast way":

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & +1 & +1 & +2 & +1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}}_{M_C^{-1} M_B} = P_{C\leftarrow B}$$

2) $p|_C = M_{C\leftarrow B}\, p|_B$,

and $p(x) = 3(1+x) + (1+x+x^2) \Rightarrow p|_B = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, so

$$p|_C = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}.$$

<u>Exercise</u>: Let $V = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$

1) Is V a vector subspace of $\mathbb{R}^3$?

<u>Sol</u>: Yes, it is. One can prove it using the two conditions (closed under addition, scaling), but notice that

$$x_1 + x_2 + x_3 = 0 \iff \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \iff$$

$$\iff A\vec{x} = \vec{0}, \text{ with } A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

That is, $\| V = N(A) \|$,

and we know that the nullspace of a matrix is a vector subspace of $\mathbb{R}^n$! (Here $\mathbb{R}^3$).

• <u>Remark</u>: All vector subspaces of $\mathbb{R}^n$ can be written as the nullspace of a matrix.

2) Find a basis for V.

From what we said in part 1), $V = N(A)$, so we have to find a basis for $N(A)$. We know how to do this:

Solve $A\vec{x} = \vec{0}$:

$$\begin{bmatrix} \boxed{1} & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \to \begin{matrix} x_2 = \alpha \\ x_3 = \beta \end{matrix} \Big| \, x_1 = -\alpha - \beta \to$$

free variables

$$\to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Basis for } V = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

<u>Remark</u>: Fast way: Notice that V is a plane in $\mathbb{R}^3$, so we know that $\dim V = 2$ (i.e., any basis of V has two vectors). So simply find two vectors satisfying the equation $x_1 + x_2 + x_3 = 0$ (that are l.i. of course).

for example $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \ldots$

3) Find the change of ~~basis~~ basis matrix from the previous basis to $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$.

Sol:

Denote $B=\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$, $C=\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$.

→ Notice that the "general method" fails here:

$$M_B=\begin{bmatrix}-1&-1\\1&0\\0&1\end{bmatrix},\ M_C=\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}\to M_C^{-1}\ ??$$

So let's do it from the start.

We want to find $P_{C\leftarrow B}$ such that $\vec{v}|_C=P_{C\leftarrow B}\vec{v}|_B$.

We have that,

$$\vec{v}=\alpha_1\vec{c}_1+\alpha_2\vec{c}_2=\beta_1\vec{b}_1+\beta_2\vec{b}_2,\quad \left(\text{where } \vec{v}|_B=\begin{bmatrix}\beta_1\\\beta_2\end{bmatrix},\ \vec{v}|_C=\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}\right)$$

that is,

$$\alpha_1\begin{bmatrix}1\\-1\\0\end{bmatrix}+\alpha_2\begin{bmatrix}0\\1\\-1\end{bmatrix}=\beta_1\begin{bmatrix}-1\\1\\0\end{bmatrix}+\beta_2\begin{bmatrix}-1\\0\\1\end{bmatrix}\to$$

$$\to\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}=\begin{bmatrix}-1&-1\\1&0\\0&1\end{bmatrix}\begin{bmatrix}\beta_1\\\beta_2\end{bmatrix}$$

Notice that these are 3 equations for 2 variables. 1 eq. should be lin. dep.!

Indeed,

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

That is,

$$\cancel{\alpha} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

In other words,

$$\underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}_{\vec{v}|_C} = \underbrace{\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}}_{P_{C\leftarrow B}} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\vec{v}|_B} \quad //.$$

Check: the first vector in C, $\vec{c}_1$, is $-\vec{b}_1$, and indeed

$$\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \checkmark$$

Exercise: (Review)

A is a 2×4 matrix with nullspace given by all vectors in the form

$$\alpha \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \ \alpha, \beta \in \mathbb{R}.$$

a) Find R, the RREF of A.

Sol: $\dim N(A) = 2 \Rightarrow \dim C(A) = n - 2 = 4 - 2 = 2$

So there are two pivots, and two free variables.

Indeed, we can see that x_2 and x_4 are our parameters:

$$\left.\begin{aligned} x_1 &= 3\alpha + 6\beta \\ x_2 &= \alpha \\ x_3 &= 2\beta \\ x_4 &= \beta \end{aligned}\right\} \rightarrow R = \overset{4}{\begin{bmatrix} \boxed{1} & -3 & 0 & -6 \\ 0 & 0 & \boxed{1} & -2 \end{bmatrix}} \, 2$$

b) Find $C(A)$.

We don't know the matrix A. But $C(A) \subseteq \mathbb{R}^2$ and we have that $\dim C(A) = 2$!

So the columns of A span all $\mathbb{R}^2 \rightarrow \underline{C(A) = \mathbb{R}^2}$

A basis of $C(A)$ is, for example, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

c) Find complete solution to $R\vec{x} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

d) Show that columns 2, 3, 4 of A are linearly dependent. Hint: find a linear combination of them that gives the zero vector.

Sol: Denote $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{bmatrix}$. Then,

$EA = R$ for some invertible matrix E, and therefore,

$\vec{r}_1 = E\vec{a}_1$, $\vec{r}_2 = E\vec{a}_2$, ..., where $\vec{r}_i$ is the column ith of R.

It is easy to see that

$-2\vec{r}_2 + 2\vec{r}_3 + \vec{r}_4 = \vec{0}$, so we also have that

$$-2(E^{-1}\vec{r}_2) + 2(E^{-1}\vec{r}_3) + (E^{-1}\vec{r}_4) = -2\vec{a}_2 + 2\vec{a}_3 + \vec{a}_4 = \vec{0}.$$