$\boxed{\text{MIDTERM 1}}$: Extra problems (solutions)

$\boxed{P1}$

1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(Important! order)

Clearly the second matrix is not invertible (columns 2 and 3 are the same, so cannot be 3 pivots).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$\boxed{P2}$

Complete solution $x = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$.

1) R is 3 by 3 and there are two free variables, so only one pivot column.

We can write, using the complete solution, that

$$
\left.\begin{aligned}
x_1 &= 4 + 2\alpha + 5\beta \\
x_2 &= \alpha \\
x_3 &= \beta
\end{aligned}\right\} \text{free variables} \rightsquigarrow
$$

$$
R = \begin{bmatrix} \boxed{1} & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad d = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}
$$

2)

$$
\left.\begin{aligned}
&A \xrightarrow{R2' = R2 - 3R1,\ R3' = R3 - 5R1} R \\
&\qquad\qquad \uparrow \\
&\qquad E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}
\end{aligned}\right\} EA = R \Rightarrow
$$

$\Rightarrow A = E^{-1}R$, where $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$, so we conclude that

$$
A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 6 & 15 \\ 5 & 10 & 25 \end{bmatrix}
$$

$$
b = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 20 \end{bmatrix}
$$

$\boxed{P3}$ $A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix}$

1) $N(A) \to$ all $x \in \mathbb{R}^4$ such that $Ax = 0$:

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \xrightarrow{R2' = R2 - 3R1}$$

$$\to \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \xrightarrow{R3' = R3 - 2R2} \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1' = R1 - R2'} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2'' = \frac{1}{2} R2'}$$

$$\to \begin{bmatrix} 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \left\{ \begin{matrix} x_1 = \alpha, \; x_4 = \beta \\ x_2 = -\beta, \; x_3 = \frac{-\beta}{2} \end{matrix} \right\}$$

$$\Rightarrow \left. \begin{matrix} x_2 + x_4 = 0 \\ x_3 + \frac{x_4}{2} = 0 \end{matrix} \right\} \text{ that is, } N(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \begin{matrix} x_2 + x_4 = 0 \\ 2x_3 + x_4 = 0 \end{matrix} \right\}.$$

$$\text{Basis for } N(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ -1/2 \\ 1 \end{bmatrix}$$

2) $C(A)$

A basis is given by the columns of A in the positions where pivots appear, so

basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix} \right\}$

Equations? By definition, a vector b is in $C(A)$ if it is a linear combination of the columns, so we need to find conditions on b such that $Ax=b$ has solutions:

$$\begin{bmatrix} 1 & 2 & \vdots & b_1 \\ 3 & 8 & \vdots & b_2 \\ 0 & 4 & \vdots & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 2 & b_2 - 3b_1 \\ 0 & 4 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 2 & b_2 - 3b_1 \\ 0 & 0 & b_3 - 2b_2 - 6b_1 \end{bmatrix} \Rightarrow$$

(*)

$$\Rightarrow C(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3 : b_3 - 2b_2 - 6b_1 = 0 \right\}$$

(*) We don't need the whole A, just the independent columns (of course!).

$\boxed{P4}$

1) The vectors $u_1, \dots, u_k$ are l.i. if the only linear combination of them giving the zero vector has all coefficients equal to zero. That is,

$u_1, \dots, u_k$ are l.i. $\iff \{ c_1 u_1 + \dots + c_k u_k = 0 \Rightarrow c_1 = \dots = c_k = 0 \}$.

2) Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}$. Is $b = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix} \in C(A)$?

We have to check that b is a linear combination of the columns of A, that is, we have to check that there exist some coefficients $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}.$$ This can be written as a system:

$$\begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}.$$ So $b \in C(A) \iff$ This system has solutions.

Let's try to solve it:

$$\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 6 & 5 & 28 \\ 2 & 4 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{array}\right] \to \left[\begin{array}{ccc} 2 & 1 & 8 \\ 0 & 2 & 4 \\ 0 & 0 & 6-4\frac{3}{2} \end{array}\right] = \left[\begin{array}{cc|c} \boxed{2} & 1 & 8 \\ 0 & \boxed{2} & 4 \\ 0 & 0 & 0 \end{array}\right] \to$$

So the system has solution $c_2 = 2 \to c_1 = (8 - c_2)/2 = 3$.

Note that this solution $c_1 = 3, c_2 = 2$ ~~are~~ gives the coefficients of the linear combination:

$$b = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}.$$

Remark: This is a longer explanation ~~that~~ than needed. I just wanted to clarify how to check if a vector is lin. dependent from others, and how to write it as a linear combination.

(not recommended to just "try looking at the vectors")

$\boxed{P5}$

1) $$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix} \xrightarrow[R3'=R3-2R1]{R2'=R2-3R1} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R3''=R3'-\frac{1}{-3}R2'} \begin{bmatrix} \boxed{1} & 2 & 1 & 4 \\ 0 & 0 & 0 & \boxed{-3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns.

rank(A) = n° of pivots = 2.

2) We continue to find the RREF of A (not really needed in this case):

$$R2''=\frac{-1}{3}R2' \rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1'=R1-4R2''} \begin{bmatrix} \boxed{1} & 2 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free variables

$x_2 = \alpha$ $\rightarrow$ $x_1 = -2\alpha - 2\beta$ $\rightarrow$

$x_3 = \beta$ $\quad$ $x_4 = 0$

$\alpha, \beta \in \mathbb{R}$.

$$\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ (Complete solution to $Ax=0$)

3) This is the same as asking for which α is $\begin{bmatrix} 3 \\ 9 \\ \alpha \end{bmatrix}$ in $C(A)$?

We check is this vector is a linear combination of the (independent) columns of A:

$$\left[\begin{array}{cc|c} 1 & 4 & 3 \\ 3 & 9 & 9 \\ 2 & 9 & \alpha \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \boxed{1} & 4 & 3 \\ 0 & \boxed{-3} & 0 \\ 0 & 1 & \alpha-6 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \boxed{1} & 4 & 3 \\ 0 & \boxed{-3} & 0 \\ 0 & 0 & \alpha-6 \end{array}\right] \rightarrow$$

$\Rightarrow \alpha = 6$ is the only value that makes the system solvable.

$\boxed{P6}$

1) Write $B = \left[b_1 \,\vdots\, b_2 \,\vdots\, b_3 \right]$, where b_1, b_2, b_3 are the columns of B.

Then,

$$AB = \left[Ab_1 \,\vdots\, Ab_2 \,\vdots\, Ab_3 \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{since } b_1, b_2, b_3 \in N(A).$$

2) For A to be invertible, it has to have 3 pivots:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & \alpha \end{bmatrix} \xrightarrow[R3' = R3 - R1]{R2' = R2 - R1} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 6 \\ 0 & 6 & \alpha - 3 \end{bmatrix} \xrightarrow{R3'' = R3' - 2R2'} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & \alpha - 3 - 18 \end{bmatrix}$$

So $\alpha - 21 \neq 0 \Rightarrow \alpha \neq 21$. For all $\alpha \neq 21$ A is invertible.

3) From steps in 2),

$$A \rightsquigarrow U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & -21 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ +1 & 1 & 0 \\ +1 & +2 & 1 \end{bmatrix} \quad (\text{Check that } A = LU)$$

I' (not needed) multipliers

$\boxed{P7}$

1)

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$ (not really needed for this).

free ↑ ↑ ↑ pivots (they aren't l.i)

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

all pivot columns.

2)

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & b_1 \\ 0 & 2 & 2 & b_2 \\ 0 & 3 & 4 & b_3 \end{array}\right] \longrightarrow \begin{bmatrix} 0 & \boxed{1} & 1 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & \boxed{1} & b_3 - 3b_1 \end{bmatrix} \longrightarrow b_2 = 2b_1.$$

$Ax=b$ has solution only for those b such that $b_2 = 2b_1$.

Complete solution to $Ax=b$: ($b_2 = 2b_1$)

$x_1 = \alpha$ (free)

$x_2 = b_1 - x_3 = b_1 - b_3 + 3b_1$

$x_3 = b_3 - 3b_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4b_1 - b_3 \\ b_3 - 3b_1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(basis for $N(A)$)

3) M has an inverse because there are three pivots:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \Rightarrow$$

$$\Rightarrow M^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ 4 & 0 & -1 \\ -3 & 0 & 1 \end{bmatrix} \quad \text{(Check that } M^{-1}M = I\text{)}$$

$\boxed{\text{P8}}$

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}}_{A} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{R3' = R3 - 4R1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2' = R2 - R3'} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{23}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_{12}} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \Rightarrow$$

$$\Rightarrow A = (E_{23} E_{31} P_{12})^{-1} I = P_{12}^{-1} E_{31}^{-1} E_{23}^{-1} =$$ (just to check the result)

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 4 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix}}_{A}$$ //.

From the previous steps it is clear that

$$A^{-1} = E_{23} E_{31} P_{12} \,//.$$

$\boxed{P9}$

1) When no row exchanges, we know that elimination produces

$A = LU$, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & g \end{bmatrix} \Rightarrow$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 9 & 6+g \end{bmatrix}$$

2) $g = 0 \rightarrow N(A)$? (Could start from here)(we have U!)

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 9 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_3 = \alpha \\ x_1 = -\alpha/2 \\ x_2 = -\alpha/2 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \rightarrow \text{Basis for } N(A) = \left\{ \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right. .$$

3) If $g \neq 0$, there are three pivots, so $N(U) = \{0\} = N(A)$.

$\boxed{\text{P10}}$ (Hard)

1) Nullspace given by $\alpha \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, i.e.,

$$\begin{aligned} x_1 &= -3\alpha - 6\beta \\ x_2 &= \alpha \\ x_3 &= -2\beta \\ x_4 &= \beta \end{aligned}$$

$\leftarrow$ free variables ($\Rightarrow$ 2 pivot columns: 1^{st} and 3^{rd}).

$$R = \begin{bmatrix} 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

1^{st} 3^{rd}
pivots

2) $C(A)$ = "columns of A in position of pivots" = $\{\ ?\ \}$
basis for

we don't know A!!.

<u>But</u> we see that there are two pivot columns, and the columns have two components! That is, $C(A)$ is a subspace of $\mathbb{R}^2$ with two vectors in its basis! Obviously, $\boxed{C(A) = \mathbb{R}^2}$

So a basis would be $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

3) $\begin{bmatrix} \boxed{1} & -3 & 0 & -6 \\ 0 & 0 & \boxed{1} & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \rightarrow \begin{matrix} x_2 = \alpha \\ x_4 = \beta \end{matrix} \rightarrow \begin{matrix} x_1 = 3 + 3\alpha + 6\beta \\ x_3 = 6 + 2\beta \end{matrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

4) Let $A = [a_1 | a_2 | a_3 | a_4]$ where $a_1, \dots, a_4$ are two-component vectors (columns of A).

We need to show that the system

$[a_2 | a_3 | a_4] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has a non-trivial solution, that is,

that there are numbers $c_1, c_2, c_3 \in \mathbb{R}$ (not all zero) such that

$c_1 a_2 + c_2 a_3 + c_3 a_4 = 0$.

Although we don't know A, we know that there exists an invertible matrix E such that

$EA = R$.

So, $E[a_1 | a_2 | a_3 | a_4] = [Ea_1 | Ea_2 | Ea_3 | Ea_4] = [r_1 | r_2 | r_3 | r_4]$

Therefore,

$$\left[a_2 \,|\, a_3 \,|\, a_4\right]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \iff E\left[a_2 \,|\, a_3 \,|\, a_4\right]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \iff$$

$$\iff \left[r_2 \,|\, r_3 \,|\, r_4\right]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \iff \begin{bmatrix} -3 & 0 & -6 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} -3 & 0 & -6 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & -2 \end{bmatrix} \Rightarrow$$

free variable

$$\Rightarrow \left.\begin{aligned} c_3 &= \alpha \\ c_1 &= -\alpha \\ c_2 &= 2\alpha \end{aligned}\right\}$$ that is, for example choosing $\alpha = 1$,

$$\| -a_1 + 2a_2 + a_3 = 0 \|$$

$\boxed{P11}$

1) $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & 10 \end{bmatrix} \xrightarrow[R3' = R3-3R1]{R2' = R2-2R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 4 & 7 \end{bmatrix} \xrightarrow{R3'' = R3' - 2R2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad A = LU \text{ (Check it)}.$$

2) $Ax = b \iff LUx = b$.

Call $y = Ux \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$ $\to x_1 = 3-1-1 = 1$, $\to x_2 = (4-2)/2 = 1$, $\to x_3 = 1$

$Ly = b \to \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 20 \end{bmatrix}$ $\to y_1 = 3$, $\to y_2 = 10 - 2y_1 = 4$, $\to y_3 = 20 - 3\cdot 3 - 2\cdot 4 = 3$

So, $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

3) $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & k \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 4 & k-3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & k-3-4 \end{bmatrix} \to \boxed{k=7}$$ makes the matrix singular.

4)

$$A \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & k-3-4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{k = 7}$$

5) $k \neq 7 \rightarrow$ makes A invertible.

$\boxed{P12}$

$$\left.\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6 \end{aligned}\right\} \quad \left[\begin{array}{cc|c} k & 3 & 6 \\ 3 & k & -6 \end{array}\right]$$

- $\underline{k=0}$: $\begin{aligned} 3y &= 6 \\ 3x &= -6 \end{aligned} \rightarrow \begin{aligned} y &= 2 \\ x &= -2 \end{aligned}$.

- $\underline{k \neq 0}$: $\left[\begin{array}{cc|c} k & 3 & 6 \\ 3 & k & -6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} k & 3 & 6 \\ 0 & k - \frac{9}{k} & -6 - \frac{18}{k} \end{array}\right]$

$\rightarrow \| \; k - \frac{9}{k} = 0 \rightsquigarrow k^2 - 9 = 0 \rightarrow k = \pm 3$:

$\underline{k=3}$: $\begin{bmatrix} 3 & 3 & 6 \\ 0 & 0 & -12 \end{bmatrix}$ No solution.

$\underline{k=-3}$: $\left[\begin{array}{cc|c} -3 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right]$ Infinite solutions.

$$\left[\begin{array}{cc|c} -3 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} y = \alpha \\ \hookrightarrow x = (6-3\alpha)/(-3) = -2+\alpha \end{array} \Big\} \rightarrow$$

$$\rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \alpha \in \mathbb{R}.$$

→ $\underline{k \neq \pm 3}$: Then,

$$y = \left(-6 - \frac{18}{k}\right) / \left(k - \frac{9}{k}\right) = \frac{-6k-18}{\cancel{k}} \cdot \frac{\cancel{k}}{k^2-9} = \frac{18+6k}{9-k^2} \rightarrow$$

$$\rightarrow x = (6-3y)/k = \frac{1}{k} \, \frac{\cancel{54} - 6k^2 - \cancel{54} - 18k}{9-k^2} = \frac{18-6k}{9-k^2}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18-6k \\ 18+6k \end{bmatrix} \frac{1}{9-k^2}.$$

$\underline{\text{Summary}}$: Solvable for all $k \neq 3$.

1) $\underline{k \neq \pm 3}$ → One solution: $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9-k^2} \begin{bmatrix} 18-6k \\ 18+6k \end{bmatrix}$.

2) $\underline{k = -3}$ → Infinite solutions $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\boxed{\text{P13}}$

1) Yes, it is a plane indeed. Let's check:

$$\left.\begin{array}{l}(x_1,y_1,z_1) \,/\, 10x_1+y_1+2018z_1=0\\ (x_2,y_2,z_2) \,/\, 10x_2+y_2+2018z_2=0\end{array}\right\}$$

· Addition: $(x_1+x_2, y_1+y_2, z_1+z_2)$ has to verify the equation:

$$10(x_1+x_2)+(y_1+y_2)+2018(z_1+z_2)=0.$$

This is clear: $10x_1+10x_2+y_1+y_2+2018z_1+2018z_2=$

$$=\underbrace{(10x_1+y_1+2018z_1)}_{=0}+\underbrace{(10x_2+y_2+2018z_2)}_{=0}=0 \checkmark.$$

· Scaling: $(\lambda x_1, \lambda y_1, \lambda z_1)$

$$\hookrightarrow 10(\lambda x_1)+(\lambda y_1)+2018(\lambda z_1)=\lambda\underbrace{(10x_1+y_1+2018z_1)}_{=0}=0 \checkmark.$$

2) No: Take $(x,y,z)=(-1,-1,-1)$.

Choose the scalar $\lambda=-1 \rightarrow \lambda(-1,-1,-1)=(1,1,1)$ but $1+1+1\neq 0$.

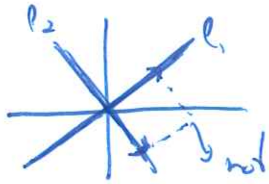
3) Yes, it is a line indeed. (basically repeat 1.).

4) No: Take $(1,-1,0)$ (satisfies $1-1+0=0$).

$(2,-1,0)$ (satisfies $(2-2+0)=0$.

However $(1,-1,0)+(2,-1,0)=(3,-2,0)$ doesn't satisfy any of the condition:
$$\left.\begin{array}{l} 3-2+0\neq 0 \\ 3-4+0\neq 0 \end{array}\right\}$$

Π Graphically: think in 2d, $\mathbb{R}^2$



intersection of l_1 and $l_2 = \{0\}$ (it is a subspace).

one line or the other: it isn't.

not in l_1, not in l_2

5) Yes, that's the definition of column space.

$\boxed{P14}$

1) Let's first find a basis for that plane:

$$x+y+z=0 \Leftrightarrow [\,1 \,\vdots\, 1 \;\; 1\,]\begin{bmatrix}x\\y\\z\end{bmatrix}=0 \to \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\alpha\begin{bmatrix}-1\\1\\0\end{bmatrix}+\beta\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

free

So for example,

$$A=\begin{bmatrix}-1 & -1 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

2) $C(A)=$ "plane" $\to \dim C(A)=2 \to$ not three pivots.

indeed, basis of $C(A)=\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$.

3) $\left.\begin{array}{l}B \text{ has to be 3 by 4, and } \dim C(B)=2,\ \dim N(B)=1\\ \text{Rank-Nullity theorem: } 4=\dim C(B)+\dim N(B)\end{array}\right\}$ Not possible.

4) No: the zero matrix has column space $\{0\}$.