1

Problem 1:

1. Note that

$$A^2 V_0 = A \cdot (A V_0) = A(0 V_0) = 0(A V_0) = 0^2 V_0$$

Similarly for V_1 and V_2, so V_0, V_1 and V_2 are the eigenvectors of A with eigenvalues $0, 1$ and $2^2=4$ respectively

2. We have

$$(A+I) V_0 = A V_0 + V_0 = 0 \cdot V_0 + V_0 = (0+1) V_0$$

Similarly for V_0, V_1 and V_2. So these are also eigenvectors of $A+I$ with eigenvalues $1, 2$ and 4 respectively.

3. A is not invertible because $A V_0 = 0$.

Problem 2

1. $$\begin{cases} a+b=6 \\ a+2b=4 \\ a+4b=0 \end{cases}$$

2. Let $X=\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$, then $\begin{bmatrix} a \\ b \end{bmatrix}$ is the solution to

$$X^T X \begin{bmatrix} a \\ b \end{bmatrix} = X^T \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

Problem 3

1. The system is

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}_{=X} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Note that the columns of X are orthogonal

because of this

2. $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the solution to $X^T X \begin{bmatrix} a \\ b \\ c \end{bmatrix} = X^T \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = X^T \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Problem 4

1. P maps $V^\perp$ to 0, so $(1,2,1)$ is an eigenvector of P w/ eigenvalue 0. $(1,0,-1)$ and $(2,-1,0)$ are in V, so $P\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $P\begin{pmatrix}2\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}$. So they are eigenvectors of P with eigenvalue 1.

2. Since $\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\0\end{pmatrix} \perp \begin{pmatrix}1\\2\\1\end{pmatrix}$ the coefficient of $\vec{u}_3$ is

$$\frac{\vec{b}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1} = 2/3$$

Now note that $\vec{b} - 2/3\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1/3\\-1/3\\1/3\end{pmatrix}$, so the coefficients of $\vec{u}_1$ and $\vec{u}_2$ are $-1/3$ and $1/3$ respectively.

3. Apply Gram-Schmidt to $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$.

4. Computations. The last column should be $\begin{bmatrix}0\\0\end{bmatrix}$.

4

Problem 5

1. Easy computations

2. Note that Q should be $\left[\vec{q}_1\ \vec{q}_2\right]$, because than

$$QQ^T\vec{v} = \vec{q}_1(\vec{q}_1 \cdot v) + \vec{q}_2(\vec{q}_2 \cdot v)$$

3. Solve $\underbrace{Q^T Q}_{=Id_2}\begin{bmatrix} x \\ y \end{bmatrix} = Q^T\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Problem 6

1. Impossible because $(1,1,1) \in col(A) \Rightarrow Null(A^T) \perp (1,1,1)$

2. Yes (e.g. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$)

3. No, that implies $(1,1,1) \in Null(A^T) \cap col(A) = \{0\}$ ↯

Problem 7

1. Computations.

2. $(\vec{b}\cdot\vec{q}_1)\vec{q}_1 + (\vec{b}\cdot\vec{q}_2)\vec{q}_2 + (\vec{b}\cdot\vec{q}_3)\vec{q}_3$

Problem 8

See solutions to HW 5

Problem 9

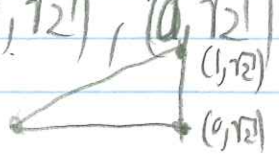
1. $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}_{\mathcal{E}}$

2. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\} = \mathcal{V}$

3. $\vec{v} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}_{\mathcal{V}}$

4. $\begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$

5. Note that the triangle is in V. The coordinates in $\mathcal{V}$ are $(0,0)$, $(1,\sqrt{2})$, $(0,\sqrt{2})$, so the area is $\det\begin{pmatrix} 1 & 0 \\ \sqrt{2} & \sqrt{2} \end{pmatrix}/2 = \sqrt{2}/2$.



Problem 10

Use the method from the book or do simple algebra to find the basis change matrix T. Then

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = T \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}.$$

Problem 11

Note that $\det(A - \lambda I) = (\lambda - 1)(\lambda + 1/9)$, so 1 and $-1/9$ are eigenvalues of A. Diagonize A:

$$A = P \begin{bmatrix} 1 & 0 \\ 0 & -1/9 \end{bmatrix} P^{-1}$$

so $$A^k = P \begin{bmatrix} 1 & 0 \\ 0 & (-1/9)^k \end{bmatrix} P^{-1} \xrightarrow{k \to \infty} P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

Problem 12

$\det(A - \lambda I) = (1-\lambda)(2-\lambda)(b-\lambda)$.

So $b \neq 1, 2$.

Problem 13

The eigenvector are $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ (actually any basis of the plane $z=0$) and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ w/ eigenvalues 1, 1 and 0.
The argumentation is the same as that of problem 1.

Problem 14:

1. $A = \begin{bmatrix}1 & 3\\2 & 2\end{bmatrix}$

2. Follow the steps in the book

3. $e^{At} = S\begin{bmatrix}e^{\lambda_1 t} & 0\\0 & e^{\lambda_2 t}\end{bmatrix}S^{-1}$ where λ_1 and λ_2 are the eigenvalues

4. $\begin{bmatrix}x(t)\\y(t)\end{bmatrix} = e^{At}\begin{bmatrix}-5\\5\end{bmatrix}$

5. Compute $\lim_{t\to\infty} y(t)/x(t)$.

Problem 15:

A: not diagonalizable because it only has one eigenvalue (3) and $A-3I \neq 0$.

B: Clearly $\begin{pmatrix}0\\5\\-3\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$ are eigenvectors w/ eigenvalues 0 and 8 respectively. Writing B in the basis $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\5\\-3\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ we have

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\1 & 0 & 8\end{bmatrix}$$

This clearly has 1 as an eigenvalue. So B has distinct eigenvalues (0, 1 and 8), so it is diagonalizable

C: $\det\begin{pmatrix}-7-\lambda & 13\\13 & 1-\lambda\end{pmatrix} = (1-\lambda)(-7-\lambda)-13^2$

$$= \lambda^2+6\lambda-176$$

This has discriminant $\Delta = 6^2-4\cdot1\cdot(-176) = 36+4\cdot176 > 0$,

so C has two distinct eigenvalues, so C is diagonalizable.

Problem 16

1. $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$,

$\det(A - \lambda I) = (2-\lambda)(-2-\lambda) + 4 = \lambda^2$

so 0 is the only eigenvector of A, but $A \neq 0$ so A is not diagonalizable

2. $A^2 = 0$, so $e^{At} = \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix}$

3. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3+6t-4t \\ 3t+1-2t \end{bmatrix} = \begin{bmatrix} 3+2t \\ 1+t \end{bmatrix}$

Problem 17

1. clearly $\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\\0\end{pmatrix}\right\}$ forms a basis

2. Comuputations

3. This is the orthogonal projection to span$\{\vec{u}_1, \vec{u}_2\}$, that is:

$$((1,1,1)\cdot\vec{u}_1)\,\vec{u}_1 + ((1,1,1)\cdot\vec{u}_2)\,\vec{u}_2$$

~~Ps~~Problem 18

Same as problem 14

True or False

1. False: $\det(2A) = 2^3\det(A) = 8$

2. False: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has ± 1 as eigenvalues, $B = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$ has eigenvalues $\frac{1}{2}(3 \pm \sqrt{13})$

3. True: If $Av = \lambda v$ then $\underbrace{A^{-1}Av}_{=v} = A^{-1}\lambda v = \lambda(A^{-1}v) \Rightarrow A^{-1}v = \frac{1}{\lambda}v$

4. True: Length of $Q\vec{x} = \sqrt{Q\vec{x} \cdot Q\vec{x}} = \sqrt{\vec{x} \cdot \underbrace{Q^TQ}_{=Id}\vec{x}} = \underbrace{\sqrt{\vec{x} \cdot \vec{x}}}_{= \text{length of } Q\vec{x}}$

$\cos(\text{Angle between } Q\vec{x} \text{ and } Q\vec{y}) = \frac{Q\vec{x} \cdot Q\vec{y}}{\|Q\vec{x}\|\|Q\vec{y}\|} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|\|\vec{y}\|} = \cos(\text{angle} \ldots)$

5. True: If A has orthogonal columns we have $A^TA = Id \Rightarrow A^T = A^{-1}$, so $AA^T = AA^{-1} = Id \Rightarrow$ rows are orthonormal

6. True: A has 0 with an eigenvalue with multiplicity 2 $\Rightarrow \dim(\text{Null}(A)) = 2 \Rightarrow \text{rank}(A) = 1$.

7. True: Let A be such a matrix, $v \in \mathbb{R}^2$, then $(Av) \cdot v = 0$ so $Av \perp v \Rightarrow Av$ is not parallel to v (and is also not 0 unless v is zero)

8. True: Let $v = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, the matrix can be written as

$\begin{bmatrix} v & 2v & v \end{bmatrix}$, so v is an eigenvector w/ eigenvalue

$v \cdot (1,2,1) = 9$

9. False: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has rank 2, but it's only eigenspace has dimension 1 (span $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$)

10. True: Null(A) is the eigenspace associated to 0

11. True: Let A be such a matrix, $\det(A - \lambda I) = (1-\lambda)^3$ so 1 is the only eigenvalue $\Rightarrow$ A diagonalizable iff $A - I = 0$.
(if, and only if)

12. False: $\det((B-I)^2 A) = \underbrace{(\det(B-I))}_{=0 \text{ because } 1 \text{ is an eigenvalue of } B}{}^2 \det(A)$

13. True: If $Av = \lambda v$, then $\|v\|^2 = Av \cdot Av = \lambda^2 \|v\| \Rightarrow \lambda^2 = 1$.

14. False: (Same as 1)

15. True: $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$ (n odd)
$\Rightarrow \det(A) = 0$.