HOMEWORK ASSIGNMENT 5

Name: **Due:** Friday March 22, 4PM

All the problems in this homework are from W. Strauss book.

For the problems corresponding to Chapter 4 (Problems 1 to 8) you don't need to compute the coefficients of the series expansions.

Problem 1: Strauss, Section 4.1 #3, p.89

A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, L)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0, L)$ with Dirichlet conditions at the ends (here i is the complex unit). Separate the variables to find the representation of its solution as a series.

Problem 2: Strauss, Section 4.2 #1, p.92

Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < L$, with the mixed boundary conditions $u(0, t) = u_x(L, t) = 0$.

Problem 3: Strauss, Section 4.2 #2, p.92

Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < L$, with boundary conditions $u_x(0, t) = u(L, t) = 0$ (Neumann at the left, Dirichlet at the right).

(a) Show that the eigenfunctions are $\cos\left((n + \frac{1}{2})\pi x/L\right)$.
(b) Write the series expansion for a solution $u(x, t)$.

Problem 4: Strauss, Section 4.2 #3, p.92

Solve the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < L$ with the boundary conditions $u_x(0, t) = 0 = u(L, t)$.

Problem 5: Strauss, Section 4.2 #4, p.92

Consider the diffusion equation inside an enclosed circular tube. Let its length (circumference) be $2L$. Let x denote the arc length parameter where $-L < x < L$. Then the concentration of the diffusing substance satisfies

$$\begin{aligned} u_t &= ku_{xx}, \qquad & -L < x < L, \\ u(-L, t) &= u(L, t), \qquad & u_x(-L, t) = u_x(L, t). \end{aligned}$$

These are called periodic boundary conditions.

(a) Show that the eigenvalues are $\lambda = (n\pi/L)^2$ for $n = 0, 1, 2, 3, \ldots$. (b) Show that the concentration is

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right) e^{-n^2\pi^2 kt/L^2}.$$

Problem 6: Strauss, Section 4.3 #1, p.100

Find the eigenvalues graphically for the boundary conditions

$$X(0) = 0, \qquad X'(L) + aX(L) = 0.$$

Assume that $a \neq 0$.

Problem 7: Strauss, Section 4.3 #2, p.100

Consider the eigenvalue problem with Robin BCs at both ends:

$$-X'' = \lambda X$$
$$X'(0) - a_0 X(0) = 0, \quad X'(L) + a_L X(L) = 0.$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_L = -a_0 a_L L$.
(b) Find the eigenfunctions corresponding to the zero eigenvalue (Hint: First solve the ODE for $X(x)$. The solutions are not sines or cosines.)

Problem 8: Strauss, Section 4.3 #9, p.100

On the interval $0 \leq x \leq 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X$$
$$X'(0) + X(0) = 0 \qquad X(1) = 0$$

(absorption at one end and zero at the other).

(a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.
(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.
(c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.
(d) Is there a negative eigenvalue?

Problem 9: Strauss, Section 5.1 #4, p.111

Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

Problem 10: Strauss, Section 5.1 #5, p.112

Given the Fourier sine series of $\phi(x) = x$ on $(0, L)$. Assume that the series can be integrated term by term, a fact that will be shown later.

(a) Find the Fourier cosine series of the function $x^2/2$. Find the constant of integration that will be the first term in the cosine series.
(b) Then by setting $x = 0$ in your results, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Problem 11: Strauss, Section 5.1 #9, p.112

Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi$, with the boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$ and the initial conditions $u(x,0) = 0$, $u_t(x,0) = \cos^2 x$.

Problem 12:

Read Chapter 5 of W. Strauss book. Which part was most confusing to you?