$\boxed{\begin{array}{l}\text{MATH 425}\\ \text{LECTURE 4}\end{array}}$: The Wave, Heat and Laplace equations.

We ended last day with the transport equation

$$\rho_t + (g(\rho))_x = 0,$$

where the flux $g(\rho)$ is given by $g(\rho) = c\rho$, assuming the flow has constant velocity c.

- The solution to this equation given certain initial data is rather simple:

$$\left.\begin{array}{l}\rho_t + c\rho_x = 0\\ \rho(x,0) = \rho_0(x)\end{array}\right\} \longrightarrow \text{Characteristics: } \frac{dx}{dt} = c \Rightarrow x(t) = ct + x_0,$$

so $\rho(x,t) = f(x-ct)$.

Since $\rho(x,0) = f(x) = \rho_0(x)$, we find that

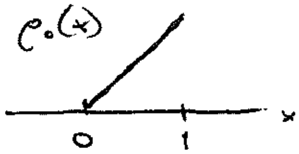
$$\boxed{\rho(x,t) = \rho_0(x-ct)}$$

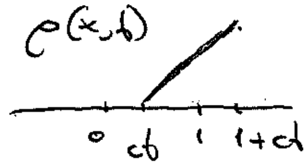
This is easier to understand with a picture:

$$\rho(x,0) = \begin{cases} x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \longrightarrow \rho(x,t) = \rho_0(x-ct) = \begin{cases} x-ct & 0 < x-ct < 1 \\ 0 & \text{otherwise} \end{cases},$$

that is,

$$\rho(x,t) = \begin{cases} x - ct & ct < x < 1+ct \\ 0 & \text{otherwise} \end{cases}$$





- A <u>nonlinear</u> example : Burger's equation.

$$u_t + u u_x = 0$$

It is like a transport equation where the velocity coincides with the density (it comes however from simplifying the fluid mechanics equations).

Let's solve it with initial data $u(x,0) = u_0(x)$:

$$\left.\begin{array}{l} u_t + u u_x = 0 \\ u(x,0) = u_0(x) \end{array}\right\}$$

Notice we can write it as $(1,u)\cdot \nabla u = 0 \Rightarrow u$ is constant along the curve defined by its tangent vector $(1,u)$.

That is, let's find the characteristics curves:

$$\frac{dx}{dt}(t) = u(x(t),t) = \underset{\uparrow}{u(x(0),0)} \Rightarrow x(t) = u(x(0),0)\,t + x_0 \Rightarrow$$

Key step: u is constant along these curves!

For a given $u_0(x)$, from $x(t) = u_0(x_0)t + x_0$, we would find

$x_0 = x_0(x,t)$ [That is, this tells us where the "fluid particle" that now is located at (x,t) was at time $t=0$].

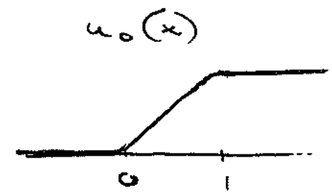
and hence,

$$u(x,t) = f(x_0) = f(x_0(x,t))$$
$$u(x,0) = u_0(x) = f(x_0(x,0)) \Rightarrow f.$$

<u>Example:</u>

1) $u_t + uu_x = 0$

$$u(x,0) = u_0(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$



Find $u(x,t)$:

$$x(t) = u_0(x_0)t + x_0 = \begin{cases} x_0 & \text{if } x_0 \le 0 \\ x_0 t + x_0 & \text{if } 0 < x_0 < 1 \\ t + x_0 & \text{if } x_0 \ge 1 \end{cases}$$

Thus,

$u\cancel{(x,t) = u_0(x_0(x,t))} =$

If $x_0 \le 0 \to u(x(t),t) = u(x_0,t) = u(x_0,0) = u_0(x_0) = 0,$

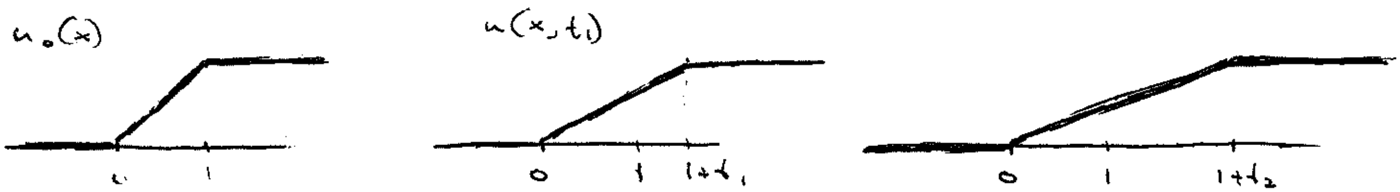
If $0 < x_0 < 1 \to u(x(t),t) = u(x_0 t + x_0, t) = u(x_0,0) = u_0(x_0) = x_0,$

If $x_0 \geq 1 \to u(x(t),t) = u(t + x_0, t) = u(x_0,0) = u_0(x_0) = 1,$

in summary,

$$u(x,t) = \begin{cases} 0 & \text{if } x \le 0 \\ \dfrac{x}{t+1} & \text{if } 0 < \dfrac{x}{t+1} < 1 \quad \to \\ 1 & \text{if } x - t \geq 1 \end{cases}$$

$$\to u(x,t) = \begin{cases} 0 & \text{if } x \le 0 \\ x/(t+1) & \text{if } 0 < x < 1+t \\ 1 & \text{if } x \geq 1+t \end{cases}$$



Exercise (Home)

1) Rarefaction wave → Question: Can you find the value $u(7,5)$? (i.e., at $x=7, t=5$)

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$$

2) Shock wave: See homework 2, Problem 5.

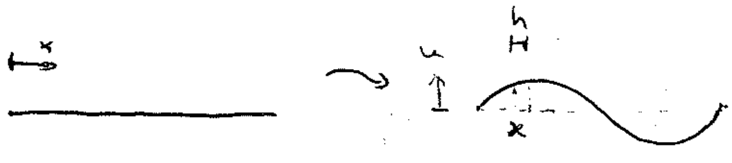
1.2.2 The wave equation.

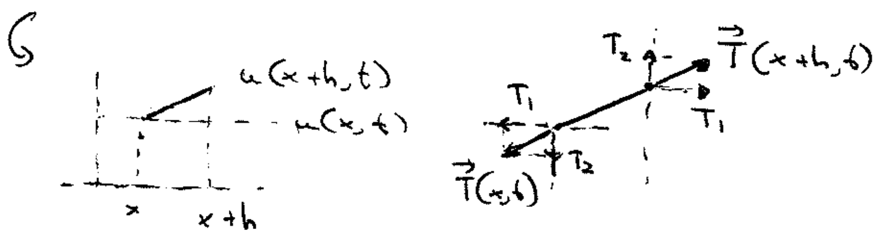
• One dimension: Vibrating string.

Consider a string moving <u>in a plane</u>. Assuming that

- It is very thin and homogeneous (density $\rho = \text{cte}$),
- Completely flexible (all forces tangent to the string),
- Small displacement,

we can apply Newton's law to obtain an equation that describes its movement.

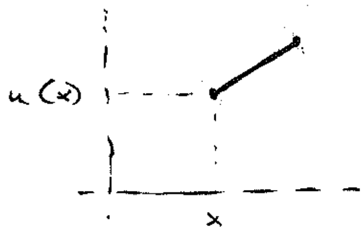




Considering a very small piece of the string (length h), we have:

$$\left.\begin{array}{ll}[H]\ \text{Horizontal} \rightarrow & 0 = T_1(x+h) - T_1(x) \\ [V]\ \text{Vertical} \rightarrow & \underbrace{\rho}_{\text{homogeneous}} h\, u_{tt}(x+h/2) = T_2(x+h) - T_2(x)\end{array}\right\}$$

What are T_1, T_2 ?



Tangent vector: $\vec{c}(x) = (1, u_x(x))$

$\downarrow$

Unit tangent vector: $\vec{s}(x) = \dfrac{\vec{c}(x)}{\|\vec{c}(x)\|} = \dfrac{(1, u_x(x))}{\sqrt{1+u_x^2(x)}}$

Thus, flexible string $\Rightarrow \vec{T}$ only in tangent direction

$$T_1(x) = \vec{T}(x)\cdot(1,0) = T(x)\,\vec{s}(x)\cdot(1,0) = T(x)\,\frac{1}{\sqrt{1+u_x^2(x)}}$$

$$T_2(x) = \vec{T}(x)\cdot(0,1) = T(x)\,\vec{s}(x)\cdot(0,1) = T(x)\,\frac{u_x(x)}{\sqrt{1+u_x^2(x)}}$$

Therefore,

$$\left.\begin{array}{l} [H] \Rightarrow T(x+h)\dfrac{1}{\sqrt{1+u_x^2(x+h)}} = T(x)\dfrac{1}{\sqrt{1+u_x^2(x)}} \\[2ex] [V] \rightarrow \rho h\, u_{tt}(x+h/2) = T(x+h)\dfrac{u_x(x+h)}{\sqrt{1+u_x^2(x+h)}} - T(x)\dfrac{u_x(x)}{\sqrt{1+u_x^2(x)}} \end{array}\right\}$$

The small displacement assumption means that

$|u_x| = \varepsilon < 1$, so we can perform a Taylor expansion as follows

$$\frac{1}{\sqrt{1+u_x^2}} = \,?$$

$$f(z) = (1+z^2)^{-1/2} \rightarrow f(z) = f(0) + f'(0)z + \frac{1}{2}f''(0)z^2 + \ldots =$$

$$= \cancel{1 - \frac{1}{2}z(1+z^2)^{-3/2}2z\Big|_{z=0}} = 1 - \frac{z^2}{2} + O(z^4)$$

That is,

$$\frac{1}{\sqrt{1+u_x^2}} = 1 - \frac{u_x^2}{2} + O(u_x^4) \approx 1$$

So we find that

$$\begin{aligned}
[H] &\Rightarrow T(x+h) = T(x) \quad \xrightarrow{x,\ h \text{ are arbitrary}} \quad T \equiv \text{cte.} \\
[V] &\Rightarrow \rho h\, u_{tt}(x+h/2) = T(x+h)\, u_x(x+h) - T(x)\, u_x(x)
\end{aligned} \Bigg\} \Rightarrow$$

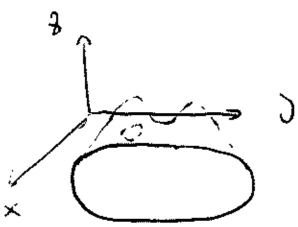
$$\Rightarrow \rho\, u_{tt}\left(x+\frac{h}{2}\right) = T\, \frac{u_x(x+h)-u_x(x)}{h} \xrightarrow{h \to 0^+}$$

$$\Rightarrow \rho\, u_{tt}(x) = T\, u_{xx}(x)$$

($T \equiv$ modulus of $\vec{T}$)
$\hookrightarrow T > 0$

In summary, $\boxed{u_{tt}(x) = c^2 u_{xx}(x)}$ $\left(c^2 = \frac{T}{\rho}\right)$

- Two dimensions: Vibrating drumhead



$\xrightarrow{\text{small piece}}$

Surface: $(x, y, u(x,y))$

$$\text{Tangents vectors: } \begin{matrix}(1,0,u_x)\\(0,1,u_y)\end{matrix} \Bigg\} \Rightarrow \text{Normal vector}: \begin{vmatrix} i & j & k \\ 1 & 0 & u_x \\ 0 & 1 & u_y \end{vmatrix} = (-u_x, -u_y, 1) \quad (*)$$

Tangent force: $\vec{T} = T_1(x,y)(1,0,u_x) + T_2(x,y)(0,1,u_y)$

Π (*) We should have written $\vec{T}$ using ~~the~~ unit tangent vectors: $\vec{T} = T_1(x,y)\dfrac{(1,0,u_x)}{\sqrt{1+u_x^2}} + T_2(x,y)\dfrac{(0,1,u_y)}{\sqrt{1+u_y^2}}$,

but again we assume small displacement, $|u_x|, |u_y| \ll 1$, so we neglect quadratic terms. $\underline{\underline{||}}$

Newton's law:

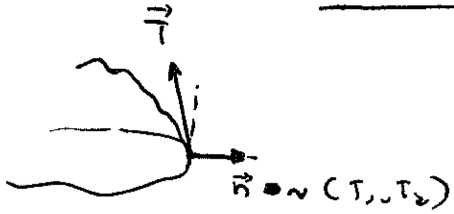
Vertical: $\displaystyle\iint_D \rho(x,y)\, u_{tt}(x,y)\,dx\,dy =$ total force in vertical direction

$$= \int_{\partial D} \vec{T}\cdot(0,0,1)\,ds,$$

$$\vec{T}\cdot(0,0,1) = \big(T_1(x,y),\, T_2(x,y),\, T_1(x,y)u_x(x,y) + T_2(x,y)u_y(x,y)\big)\cdot(0,0,1) =$$

$$= T_1 u_x + T_2 u_y = (T_1, T_2)\cdot \nabla u$$

Now, we notice that (T_1, T_2) are the first two components of $\vec{T}$ (i.e., $\vec{T}$ minus the vertical component) and thus it is the same as the normal vector to ∂D:



We can write $(T_1, T_2) = T\,\vec{n}$, where

$\vec{n} \equiv$ unit normal to ∂D

$T^2 = T_1^2 + T_2^2$

Therefore,

$$\iint_D \rho\, u_{tt}\, dx\,dy = \int_{\partial D} T\, \vec{n}\cdot\nabla u\, ds = \iint_D \nabla\cdot(T\,\nabla u)\, dx\,dy$$

Green's theorem

$$\left(\iint_D \nabla\cdot\vec{J}\, dx\,dy = \int_{\partial D} \vec{J}\cdot\vec{n}\, ds \right)$$

Finally, since T is constant (*) and D arbitrary,

$$u_{tt} = c^2\, \nabla\cdot\nabla u = c^2 \Delta u, \quad \text{where} \quad c^2 = \frac{T}{\rho}, \quad \Delta u = u_{xx} + u_{yy}.$$

(*) That T (modulus of $\vec{T}$) is constant has to be deduced from the balances (Newton's law) in the x-y directions.

- Summary: The wave equation in n dimensions is

$$\| \; u_{tt} = c^2 \Delta u \; \| \quad , \text{where} \quad \Delta u = \sum_{i=1}^{n} u_{x_i x_i}$$

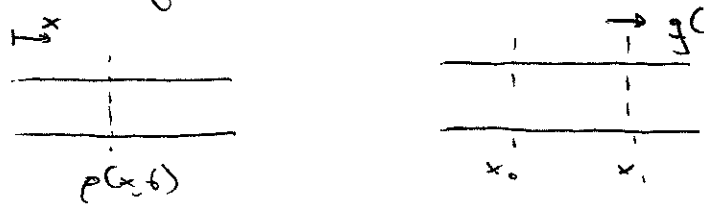
WAVE EQUATION

1.2.3 | The heat equation.

1) Diffusion

Think of a motionless fluid with some substance in it.

Denote $\rho(x,t)$ the density of this substance.

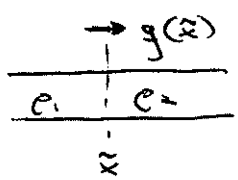


While the fluid is not moving itself, the substance will move from where its concentration (density) is higher to where it is lower (i.e. it will "diffuse").

In particular, the general model for this is Fick's law:

$g(x) = -k\,\rho_x(x)$ (flux at x) $(k>0)$.

<u>Example:</u>



- If $\rho_1 < \rho_2 \Rightarrow \rho_x(\tilde{x}) > 0 \Rightarrow g(\tilde{x}) = -k\,\rho_x(\tilde{x}) < 0 \Rightarrow$

  $\Rightarrow$ The flux goes in the opposite direction, that is, from right to left (since $\rho_2 > \rho_1$).

Mass conservation:

$$\underbrace{\int_{x_0}^{x_1} \rho(x,t+h)\,dx}_{\substack{\text{Mass at}\\ t+h}} = \underbrace{\int_{x_0}^{x_1} \rho(x,t)\,dx}_{\substack{\text{Mass at}\\ t}} + \underbrace{\int_t^{t+h} g(x_0,\tau)\,d\tau}_{\substack{\text{flux "in"}\\ \text{at } x_0}} - \underbrace{\int_t^{t+h} g(x_1,\tau)\,d\tau}_{\substack{\text{flux "out"}\\ \text{at } x_1}}$$

<u>Remark</u>: Although I wrote flux in/out, it will depend on the sign of $g(x_0,\tau)$, $g(x_1,\tau)$.

Thus,

$$\int_{x_0}^{x_1} \frac{\rho(x,t+h)-\rho(x,t)}{h}\,dx = \frac{-1}{h}\int_t^{t+h} k\,\rho_x(x_0,\tau)\,d\tau - \frac{-1}{h}\int_t^{t+h} k\,\rho_x(x_1,\tau)\,d\tau \xrightarrow{h\to 0^+}$$

$$\Rightarrow \int_{x_0}^{x_1} \rho_t(x,t)\,dx = -k\,\rho_x(x_0,t) + k\,\rho_x(x_1,t) \underset{(*)}{\xrightarrow{\partial x_1}}$$

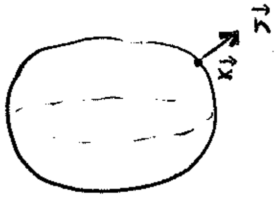
$$\Rightarrow \rho_t(x_1,t)\,dx = k\,\rho_{xx}(x_1,t) \xrightarrow{(x_1 \text{ is arbitrary})}$$

$$\| \rho_t = k\,\rho_{xx} \| \quad \text{Heat equation} \quad (k>0)$$

[(*) Also: $-k\,\rho_x(x_0,t) + k\,\rho_x(x_1,t) = k\int_{x_0}^{x_1} \rho_{xx}(x,t)\,dx \ldots$]

• In higher dimensions,

Fick's law: $g(\vec{x}) = \vec{n}(\vec{x}) \cdot \nabla \rho(\vec{x})\, k = k \dfrac{d\rho}{d\vec{n}}(\vec{x})$



Ex: If $\rho_{inside} > \rho_{outside}$,

$g(\vec{x}) = k\, \vec{n} \cdot \nabla \rho(\vec{x}) > 0$ → which means the diffusion happens in the direction of $\vec{n}$.

Mass conservation:

$$\iiint_D \rho_t(\vec{x},t)\, dx\,dy\,dz = \iint_{\partial D} g(\vec{x})\, dS = \iint_{\partial D} k\, \vec{n} \cdot \nabla \rho \, dS \underset{\text{Gauss Theorem}}{=}$$

$$= \iiint_D \nabla \cdot (k \nabla \rho)\, dx\,dy\,dz$$

As D is arbitrary,

$$\| \rho_t = k \Delta \rho \| \quad (k>0) \quad \text{HEAT EQUATION}$$

→ This equation also models the temperature evolution when heat transfer is due to diffusion (from high temperatures to low ones)