Name: ____________________ **PennID**: ____________

Math 425/AMCS 525
Midterm 1
February 14, 2019

Please *turn off and put away all electronic devices.* You are allowed to use an index card (3x5") with hand-written notes on both sides during this exam. No calculators, no books. Read the problems carefully. **Show all work** (answers without proper justification will not receive full credit). Be as organized as possible: illegible work will not be graded.

Please sign and date the pledge below to comply with the Code of Academic Integrity. Don't forget to write your Name and PennID on the top of this page. Good luck!

#	Points possible	Your score
1	20	
2	20	
3	20	
4	20	
5	15	
6	5	
Total	100	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Signature Date

Problem 1 (20 pts)

Solve the following initial boundary value problem

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= 0, \\ u(x,0) &= x^2, \\ u_t(x,0) &= e^x. \end{aligned}$$

Solution (Problem 1):

Solution (Problem 1):

Problem 2 (20 pts)

Part a. [10 pts] Solve the following second-order ODE boundary-value problem:

$$\begin{aligned} u'' + \frac{1}{x}u' &= x^2, \quad 1 \le x \le 2, \\ u(1) &= 1, \\ u(2) &= 1. \end{aligned}$$

Part b. [5 pts] Denote by r the radial variable in polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$; notice that $x^2 + y^2 = r^2$). Using the chain rule, show that for radial functions the Laplace operator in 2D, $\Delta u = u_{xx} + u_{yy}$, is given in polar coordinates by

$$\Delta u = \frac{1}{r}\left(ru_r\right)_r = u_{rr} + \frac{1}{r}u_r.$$

Part c. [5 pts] Find a solution to the boundary value problem:

$$\begin{aligned} u_{xx} + u_{yy} &= x^2 + y^2, \quad 1 \le x^2 + y^2 \le 4, \\ u(x, y) &= 1 \text{ on } x^2 + y^2 = 1, \\ u(x, y) &= 1 \text{ on } x^2 + y^2 = 4. \end{aligned}$$

Hint: Use polar coordinates and look for solutions $u(r, \theta) = u(r)$, i.e., independent of θ (thus, $u_\theta = 0$). You might use parts a) and b).

Part d. (Extra credit: Optional) [5 pts] Show that the BVP in part c) has a unique solution. Thus, the solution in part c) is *the* solution.

Hint: You might need to use the integration by parts formula

$$\int_D u(x, y)\Delta u(x, y)dxdy = -\int_D |\nabla u(x, y)|^2 dxdy + \int_{\partial D} u(s)\vec{n}(s) \cdot \nabla u(s)ds,$$

where D is a bounded domain, ∂D its boundary and $\vec{n}$ the outward normal vector.

Solution (Problem 2):

Solution (Problem 2):

Solution (Problem 2):

Problem 3 (20 pts)

Solve

$$\begin{aligned} \sin(y)u_x + 2u_y + u &= 1, \\ u(0, y) &= \cos(y), \end{aligned}$$

and find the region of the xy plane in which the solution is uniquely determined.

Solution (Problem 3):

Solution (Problem 3):

Problem 4 (20 pts)

Consider the diffusion equaition $u_t = u_{xx}$ in $0 \le x \le 1, t \ge 0$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

Part a. [5 pts] Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

Part b. [7] Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.

Part c. [8 pts] Use the energy method to show that $\int_0^1 (u(x,t)^2)dx$ is a *strictly* decreasing function of t.

Solution (Problem 4):

Solution (Problem 4):

Problem 5 (15 pts)

Suppose $u(x,t)$ satisfies the following problem:

$$\begin{aligned} &u_t = u_{xx} - 3u, \qquad \text{for } 0 < x < 1, t > 0, \\ &u(x,0) = x(1-x), \qquad \text{for } 0 < x < 1 \\ &u(0,t) = u(1,t) = 0, \qquad \text{for } t > 0. \end{aligned}$$

Define the total heat energy by

$$E(t) = \frac{1}{2}\int_0^1 u^2(x,t)dx.$$

Part a. [7.5] Show that $E(t)$ is a non-increasing function of t.

Part b. [7.5] Show that

$$\lim_{t\to\infty} E(t) = 0.$$

(*Hint for part (b):* Show that $\frac{dE}{dt} \le -6E$, then solve this differential inequality).

Solution (Problem 5):

Solution (Problem 5):

Problem 6 (5 pts) Choose only one of the following three questions (if you do a second one, it will count as extra credit. A third one would be disregarded)

Option 1. [5 pts] Give a proof of the weak maximum principle for the Laplace equation on a rectangle:

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b.$$

The weak maximum principle here says: if $u(x, y)$ is a solution, then the maximum of $u(x, y)$ on the whole rectangle $0 \le x \le a$, $0 \le y \le b$ is equal to the maximum of $u(x, y)$ on the boundaries (i.e., on $x = 0$ or $x = a$ or $y = 0$ or $y = b$).

Hint: Proceed as in the heat equation, by defining an auxiliary function $v(x, y) = u(x, y) + \varepsilon|x|^2 = u(x, y) + \varepsilon(x^2 + y^2)$ $(\varepsilon > 0)$.

Option 2. [5 pts] Consider a fluid flowing along an horizontal pipe of fixed cross section in the positive x direction, with *variable velocity* $c(x)$. If a substance is suspended in the water, with $u(x, t)$ its concentration at time t, deduce from the physical principle of mass conservation the partial differential equation that models the evolution of $u(x, t)$.

Option 3. [5 pts] Solve the Burger's equations $u_t + uu_x = 0$ with initial data

$$u(x, 0) = \phi(x) = \begin{cases} 1 \text{ if } x < 0, \\ 2 \text{ if } x > 0. \end{cases} \tag{1}$$

Where is your solution uniquely defined?

Solution (Problem 6):

Solution (Problem 6):

Extra paper

Extra paper

Extra paper

Extra paper