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Math 425/AMCS 525
Midterm 1
February 14, 2019

Please *turn off and put away all electronic devices.* You are allowed to use an index card (3x5") with hand-written notes on both sides during this exam. No calculators, no books. Read the problems carefully. **Show all work** (answers without proper justification will not receive full credit). Be as organized as possible: illegible work will not be graded.

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#	Points possible	Your score
1	20	
2	20	
3	20	
4	20	
5	15	
6	5	
Total	100	

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Problem 1 (20 pts)

Solve the following initial boundary value problem

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= 0, \\ u(x,0) &= x^2, \\ u_t(x,0) &= e^x. \end{aligned}$$

Solution (Problem 1):

- Factor the operator: (*)

$$(\partial_{xx} - 3\partial_{xt} - 4\partial_{tt})u = (\partial_x - 4\partial_t)(\partial_x + \partial_t)u.$$

- Change of variables: (**)

$$\left.\begin{array}{l} \eta = 4x + t \\ \xi = -x + t \end{array}\right| \Rightarrow \left.\begin{array}{l} \partial_x = 4\partial_\eta - \partial_\xi \\ \partial_t = \partial_\eta + \partial_\xi \end{array}\right| \Rightarrow \begin{array}{l} \partial_x - 4\partial_t = -5\partial_\xi \\ \partial_x + \partial_t = 5\partial_\eta \end{array} \Bigg\|$$

- The PDE becomes $-25\,\partial_\xi\partial_\eta u = 0 \Rightarrow$

$$\Rightarrow \partial_\eta u(\xi,\eta) = c(\eta) \Rightarrow u(\xi,\eta) = f(\eta) + g(\xi) \quad \left(\text{with } f(\eta) = \int c(\eta)\,d\eta\right).$$

- Go back to x, t:

$$u(x,t) = f(4x+t) + g(-x+t) \quad \Big\| \text{ General solution.}$$

- Finally we have to impose the initial conditions. We follow the same procedure as we did to obtain D'Alembert formula:

Solution (Problem 1):

$$\left.\begin{aligned} u(x,0) &= f(4x) + g(-x) = x^2 \\ u_t(x,0) &= f'(4x) + g'(-x) = e^x \end{aligned}\right\} \to \left.\begin{aligned} 2x &= 4 f'(4x) - g'(-x) \\ e^x &= f'(4x) + g'(-x) \end{aligned}\right\} \to$$

$$\to \left.\begin{aligned} f'(4x) &= \frac{2}{5}x + \frac{1}{5}e^x \\ g'(-x) &= \frac{-2}{5}x + \frac{4}{5}e^x \end{aligned}\right\} \to \left.\begin{aligned} f(4x) &= \frac{4}{5}x^2 + \frac{4}{5}e^x + A \\ g(-x) &= \frac{x^2}{5} - \frac{4}{5}e^x + B \end{aligned}\right| \to$$

$$\to f(4x) + g(-x) = x^2 + A + B = x^2 \to A + B = 0.$$

Therefore,

$$f(x) = \frac{x^2}{4 \cdot 5} + \frac{4}{5}e^{x/4} + A,$$

$$g(x) = \frac{x^2}{5} - \frac{4}{5}e^{-x} + B, \quad \text{and thus,}$$

$$u(x,t) = \frac{(4x+t)^2}{4 \cdot 5} + \frac{4}{5}e^{x+t/4} + \frac{(-x+t)^2}{5} - \frac{4}{5}e^{x-t} =$$

$$= \frac{4}{5}\left(e^{x+t/4} - e^{x-t}\right) + x^2 + \frac{t^2}{4}.$$

(*) If you don't see it: $\partial_{xx} - 3\partial_{xt} - 4\partial_{tt} = (\partial_x + a\partial_t)(\partial_x + b\partial_t) =$
$= \partial_{xx} + (a+b)\partial_{xt} + ab\,\partial_{tt} \to$ Solve $a + b = -3$, $ab = -4$.

(**) We choose as new variables the orthogonal to $(1,-4)$, $(1,1)$.

The change $\left.\begin{aligned} \eta &= x + t/4 \\ \xi &= x - t \end{aligned}\right\}$ is analogous (and make the computations slightly easier).

Problem 2 (20 pts)

Part a. [10 pts] Solve the following second-order ODE boundary-value problem:

$$\begin{aligned} u'' + \frac{1}{x}u' &= x^2, \quad 1 \leq x \leq 2, \\ u(1) &= 1, \\ u(2) &= 1. \end{aligned}$$

Part b. [5 pts] Denote by r the radial variable in polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$; notice that $x^2 + y^2 = r^2$). Using the chain rule, show that for radial functions the Laplace operator in 2D, $\Delta u = u_{xx} + u_{yy}$, is given in polar coordinates by

$$\Delta u = \frac{1}{r}\left(r u_r\right)_r = u_{rr} + \frac{1}{r}u_r.$$

Part c. [5 pts] Find a solution to the boundary value problem:

$$\begin{aligned} u_{xx} + u_{yy} &= x^2 + y^2, \quad 1 \leq x^2 + y^2 \leq 4, \\ u(x,y) &= 1 \text{ on } x^2 + y^2 = 1, \\ u(x,y) &= 1 \text{ on } x^2 + y^2 = 4. \end{aligned}$$

Hint: Use polar coordinates and look for solutions $u(r,\theta) = u(r)$, i.e., independent of θ (thus, $u_\theta = 0$). You might use parts a) and b).

Part d. (Extra credit: Optional) [5 pts] Show that the BVP in part c) has a unique solution. Thus, the solution in part c) is *the* solution.

Hint: You might need to use the integration by parts formula

$$\int_D u(x,y)\Delta u(x,y)dxdy = -\int_D |\nabla u(x,y)|^2 dxdy + \int_{\partial D} u(s)\vec{n}(s)\cdot\nabla u(s)ds,$$

where D is a bounded domain, ∂D its boundary and $\vec{n}$ the outward normal vector.

Solution (Problem 2):

Solution (Problem 2):

Part a.): Denote $v = u' \to v' + \frac{1}{x} v = x^2 \to$

$$\to v = e^{-\int \frac{dx}{x}} \int e^{\int \frac{dx}{x}} x^2 \, dx = \frac{1}{x} \int x^3 \, dx = \frac{x^3}{4} + \frac{c_1}{x}.$$

Then,

$$u' = \frac{x^3}{4} + \frac{c_1}{x} \to u = \frac{x^4}{16} + c_1 \ln x + c_2.$$

- Boundary conditions: $u(1) = 1 = \frac{1}{16} + c_2 \to c_2 = \frac{15}{16}$.

$$u(2) = 1 = 1 + c_1 \ln 2 + \frac{15}{16} \to c_1 = \frac{-15}{16 \ln 2}$$

Thus, $\boxed{u(x) = \frac{x^4}{16} - \frac{15}{16 \ln 2} \ln(x) + \frac{15}{16}}$.

(**Remark**: You can also use b) to write $u'' + \frac{1}{x} u' = \frac{1}{x}(x u')'$.
Or simply multiply by x in $v' + \frac{1}{x} v \rightsquigarrow x v' + v = (x v)'$)

Part b): $r^2 = x^2 + y^2 \to 2r \frac{\partial r}{\partial x} = 2x \to \frac{\partial r}{\partial x} = \frac{x}{r}$ $\left(\text{also } \frac{\partial r}{\partial y} = \frac{y}{r} \right)$.

Then, $u_x \overset{(*)}{=} u_r \frac{\partial r}{\partial x} = \frac{x}{r} u_r \to u_{xx} = \frac{1}{r} u_r - \frac{x}{r^2} \frac{x}{r} u_r + \frac{x}{r} u_{rr} \frac{x}{r} =$

$$= \frac{1}{r} u_r - \frac{x^2}{r^3} u_r + \frac{x^2}{r^2} u_{rr}.$$

Similarly for y: $u_{yy} = \frac{1}{r} u_r - \frac{y^2}{r^3} u_r + \frac{y^2}{r^2} u_{rr}$.

$$\to u_{xx} + u_{yy} = \frac{2}{r} u_r - \frac{x^2 + y^2}{r^3} u_r + \frac{x^2 + y^2}{r^2} u_{rr} = u_{rr} + \frac{1}{r} u_r \,//.$$

(*) We use that u does not depend on θ ($u = u(r) \equiv$ radial).

Solution (Problem 2):

Part c): Using part b), we can write this PDE as an ODE:

$$\left.\begin{array}{l} u_{rr} + \frac{1}{r} u_r = r^2, \quad 1 \le r \le 2 \\ u(r) = 1 \quad \text{on } r = 1 \\ u(r) = 1 \quad \text{on } r = 2 \end{array}\right\} \text{This is exactly the BVP in part a).}$$

So, $u(r) = \frac{r^4}{16} - \frac{15}{16 \ln 2} \ln(r) + \frac{15}{16} \Rightarrow$

$$\Rightarrow \boxed{u(x,y) = \frac{(x^2+y^2)^2}{16} - \frac{15/2}{16 \ln 2} \ln(x^2+y^2) + \frac{15}{16}}$$

Part d: By contradiction, let u_1, u_2 be two solutions, and define $v = u_1 - u_2$. Then, $\Delta v = 0$, $v(x,y) = 0$ on $\left.\begin{array}{l} x^2+y^2=1 \\ x^2+y^2=4 \end{array}\right\}$

Denote $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4\}$.

Then,

$$v(x,y)\Delta v(x,y) = 0 \Rightarrow \int_D v(x,y)\Delta v(x,y)\,dx\,dy = 0 \overset{\text{integ. by parts}}{=}$$

$$= -\int_D |\nabla v(x,y)|^2\,dx\,dy + \int_{\partial D} \underbrace{v(s)}_{=0} \vec{n}(s) \cdot \nabla v(s)\,ds \overset{(*)}{\Rightarrow} \int_D |\nabla v(x,y)|^2\,dx\,dy = 0 \Rightarrow$$

$\Rightarrow |\nabla v(x,y)| = 0 \Rightarrow \nabla v(x,y) = 0 \quad \forall (x,y) \in D \Rightarrow v(x,y) = \text{cte in } D \Rightarrow$

$\Rightarrow v(x,y) = 0 \Rightarrow u_1 \equiv u_2$ ⨳

in particular we know it is zero on the circles $x^2+y^2=1$, $x^2+y^2=4$

(*) $= 0$ since ∂D are the two circles $x^2+y^2=1$, $x^2+y^2=4$, where $v \equiv 0$.

Problem 3 (20 pts)

Solve

$$\sin(y)u_x + 2u_y + u = 1,$$
$$u(0,y) = \cos(y),$$

and find the region of the xy plane in which the solution is uniquely determined.

Solution (Problem 3):

- Parametrize the characteristic curves by $(x(t), y(t))$, and define them by

$$\left.\begin{aligned} x'(t) &= \sin(y(t)) \\ y'(t) &= 2 \end{aligned}\right\} \rightarrow y(t) = 2t + c_1 \longrightarrow x'(t) = \sin(2t + c_1) \rightarrow x(t) = \frac{-1}{2}\cos(2t + c_1) + c_2$$

- Then, the equation can be written as an ODE along those curves:

$$\frac{d}{dt}\Big(u(x(t), y(t))\Big) = x'(t)\, u_x(x(t), y(t)) + y'(t)\, u_y(x(t), y(t)) \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} & u'(t) + u(t) = 1 \\ & u(0) = u(x(0), y(0)) = u\left(\frac{-1}{2}\cos(c_1) + c_2, c_1\right) \end{aligned}\right\}$$

Let's choose $\left.\begin{aligned} c_2 &= \frac{1}{2}\cos(c) \\ c_1 &= c \end{aligned}\right\}$ so that $u(x(0), y(0)) = u(0, c)$.

This way we find that:

Solution (Problem 3):

$$\left.\begin{array}{l} u'(t) + u(t) = 1 \\ u(0) = \cos(c) \end{array}\right\} \Rightarrow u(t) = e^{-t}\int e^{t}\,dt = e^{-t}\left(e^{t} + c_3\right) =$$
$$= 1 + c_3 e^{-t},$$

$u(0) = 1 + c_3 = \cos(c) \Rightarrow c_3 = -1 + \cos(c)$.

- In conclusion, $u(t) = 1 + (-1 + \cos(c))e^{-t}$.

- We need to go back to the variables x, y:

$$\left.\begin{array}{l} x = \frac{-1}{2}\cos(y) + \frac{1}{2}\cos(c) \\ y = 2t + c \end{array}\right\} \Rightarrow \left.\begin{array}{l} \cos(c) = 2x + \cos(y) \\ t = \frac{y}{2} - \frac{c}{2} \end{array}\right\} \Rightarrow$$

$$\overset{(*)}{\Rightarrow} \left\| \begin{array}{l} c = \arccos(2x + \cos(y)) \\ t = \frac{y}{2} - \frac{1}{2}\arccos(2x + \cos(y)). \end{array}\right.$$

Therefore, $\boxed{u(x,y) = 1 + (2x + \cos(y) - 1)e^{-\frac{y}{2} + \frac{1}{2}\arccos(2x + \cos(y))}}$.

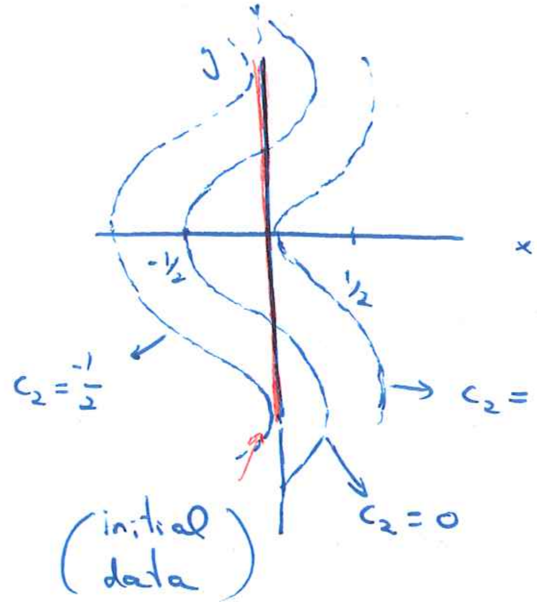
→ Region: Notice that to obtain $(*)$, it is needed that

$$|2x + \cos(y)| \leq 1.$$

That is, the region is $\{(x,y) \in \mathbb{R}^2 : -1 \leq 2x + \cos(y) \leq 1\}$.

Remark: This can also be seen by drawing the characteristic curves: ↓

$\rightarrow \quad x = \frac{-1}{2}\cos(y) + c_2$



We see that if $c_2 > \frac{1}{2}$ or $c_2 < \frac{-1}{2}$, the curves don't intersect with the line $x = 0$ (along which the initial data is given).

Thus the condition is

$$|c_2| \leq \frac{1}{2} \rightarrow \left|x + \frac{1}{2}\cos(y)\right| \leq \frac{1}{2}.$$

Problem 4 (20 pts)

Consider the diffusion equaition $u_t = u_{xx}$ in $0 \le x \le 1, t \ge 0$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.

Part a. [5 pts] Show that $0 < u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$.

Part b. [7] Show that $u(x,t) = u(1-x,t)$ for all $t \ge 0$ and $0 \le x \le 1$.

Part c. [8 pts] Use the energy method to show that $\int_0^1 (u(x,t)^2)dx$ is a *strictly* decreasing function of t.

Solution (Problem 4):

See solutions to Homework 3: Problem 8.

Solution (Problem 4):

Problem 5 (15 pts)

Suppose $u(x,t)$ satisfies the following problem:

$$\begin{aligned} &u_t = u_{xx} - 3u, && \text{for } 0 < x < 1, t > 0, \\ &u(x,0) = x(1-x), && \text{for } 0 < x < 1 \\ &u(0,t) = u(1,t) = 0, && \text{for } t > 0. \end{aligned}$$

Define the total heat energy by

$$E(t) = \frac{1}{2}\int_0^1 u^2(x,t)dx.$$

Part a. [7.5] Show that $E(t)$ is a non-increasing function of t.

Part b. [7.5] Show that

$$\lim_{t\to\infty} E(t) = 0.$$

(*Hint for part (b):* Show that $\frac{dE}{dt} \le -6E$, then solve this differential inequality).

Solution (Problem 5):

See Solutions to Practice Exam: Problem 5.

Solution (Problem 5):

Problem 6 (5 pts) Choose only one of the following three questions (if you do a second one, it will count as extra credit. A third one would be disregarded)

Option 1. [5 pts] Give a proof of the weak maximum principle for the Laplace equation on a rectangle:

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b.$$

The weak maximum principle here says: if $u(x,y)$ is a solution, then the maximum of $u(x,y)$ on the whole rectangle $0 \le x \le a$, $0 \le y \le b$ is equal to the maximum of $u(x,y)$ on the boundaries (i.e., on $x = 0$ or $x = a$ or $y = 0$ or $y = b$).

Hint: Proceed as in the heat equation, by defining an auxiliary function $v(x,y) = u(x,y) + \varepsilon|x|^2 = u(x,y) + \varepsilon(x^2 + y^2)$ $(\varepsilon > 0)$.

Option 2. [5 pts] Consider a fluid flowing along an horizontal pipe of fixed cross section in the positive x direction, with *variable velocity* $c(x)$. If a substance is suspended in the water, with $u(x,t)$ its concentration at time t, deduce from the physical principle of mass conservation the partial differential equation that models the evolution of $u(x,t)$.

Option 3. [5 pts] Solve the Burger's equations $u_t + uu_x = 0$ with initial data

$$u(x,0) = \phi(x) = \begin{cases} 1 \text{ if } x < 0, \\ 2 \text{ if } x > 0. \end{cases} \tag{1}$$

Where is your solution uniquely defined?

Solution (Problem 6):

• Option 1: Denote $R = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le a,\ 0 \le y \le b\}$, and

$T = \{(x,y) \in \mathbb{R}^2 : x = 0 \text{ or } x = a \text{ or } y = 0 \text{ or } y = b\}$.

We want to show that $\max\limits_{(x,y)\in R} \{u(x,y)\} = \max\limits_{(x,y)\in T} \{u(x,y)\}$.

Denote $M = \max\limits_{(x,y)\in T} \{u(x,y)\}$, and define $v(x,y) = u(x,y) + \varepsilon|x|^2$ $(\varepsilon > 0)$.

Then:

→ $\Delta v = \underbrace{\cancel{\Delta} v}_{=0} + \varepsilon\Delta(x^2+y^2) = 2\varepsilon > 0$, that is, $\| \Delta v > 0 \|$.

→ Thus v attains its maximum on T. Why?
↓

Solution (Problem 6):

If v attains maximum on the interior, then $\Delta v \leq 0$ $\cancel{+}$.
(and v has to attain its maximum somewhere as it's a continuous function on a bounded closed domain).

Therefore,

$$v(x,y) \leq \max_{(x,y)\in\Gamma} \{v(x,y)\} \leq \max_{(x,y)\in\Gamma} \{u(x,y)\} + \varepsilon(a^2+b^2) = M + \varepsilon(a^2+b^2).$$

This implies that

$$u(x,y) \leq M + \varepsilon(a^2+b^2-x^2-y^2) \quad \forall (x,y)\in\mathcal{R}.$$

Since $\varepsilon > 0$ was arbitrary we conclude that $u(x,y) \leq M$ $\forall (x,y)\in\mathcal{R}$, as we wanted to prove //.

• <u>Option 2:</u>

Basically, follow what we did in class to deduce the transport equation in terms of the flux:

$u_t + (g(u))_x = 0$, where the flux was $\underset{\text{velocity}}{c} \cdot \underset{\text{density}}{u}$.

Here,

$c = c(x)$, so $(g(u))_x = (c(x)u)_x = c'(x)u + c(x)u_x$.

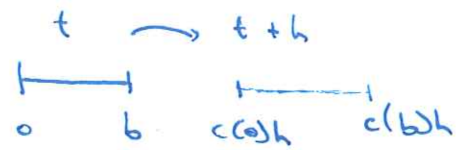
Thus, $\| u_t + c'(x)u + c(x)u = 0 \|$.

Extra paper

→ Also: (as in the book)

The amount of substance in the interval $[0,b]$ at time t is $M = \int_0^b u(x,t)\,dx$. At a later time $t+h$, each point moves $c \cdot h$ to the right (where c depends on the point):

$$M = \int_0^b u(x,t)\,dx = \int_{c(0)h}^{b+c(b)h} u(x,t+h)\,dx$$



Differentiate with respect to b:

$$u(b,t) = u(b+c(b)h,\, t+h)\,(1+c'(b)h).$$

Differentiate with respect to h and let $h \to 0$:

$$0 = \Big[u_x(b+c(b)h,\, t+h)\,c(b) + u_t(b+c(b)h,\, t+h)\Big](1+c'(b)h) +$$

$$+\, u(b+c(b)h,\, t+h)\,c'(b) \Rightarrow$$

$$\overset{h\to 0^+}{\Rightarrow} 0 = \big(u_x(b,t)\,c(b) + u_t(b,t)\big) + u(b,t)\,c'(b) \overset{b \text{ arbitrary}}{\Rightarrow}$$

$$\| u_t + c'(x)u + c(x)u_x = 0 \|$$

Extra paper

• <u>Option 3</u>: (This is the case of a "rarefaction" wave)

Let $x'(t) = u(t, x(t))$. Then,

$$\frac{d}{dt}\left(u(t, x(t))\right) = u_t(t, x(t)) + u_x(t, x(t))\, u(t, x(t)) = 0 \Rightarrow$$

$$\Rightarrow \boxed{u(t, x(t)) = u(0, x(0)) = \phi(x(0)) = \begin{cases} 1 & \text{if } x(0) < 0. \\ 2 & \text{if } x(0) > 0. \end{cases}}$$

We can now write the characteristic lines:

$$x'(t) = u(t, x(t)) = u(0, x(0)) = \phi(x(0)) \Rightarrow$$

$$\Rightarrow x(t) = \phi(x(0))\, t + x(0) \Rightarrow$$

$$\Rightarrow x(t) = \begin{cases} t + x(0) & \text{if } x(0) < 0, \\ 2t + x(0) & \text{if } x(0) > 0. \end{cases}$$

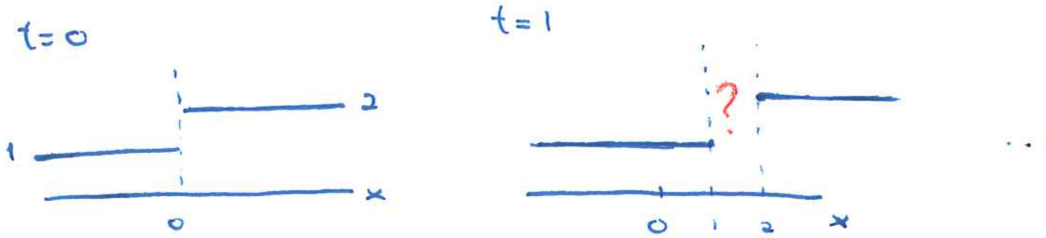
Finally,

$$\left.\begin{array}{l} u(t, t + x(0)) = 1 \quad (x(0) < 0) \\ u(t, 2t + x(0)) = 2 \quad (x(0) > 0) \end{array}\right\} \Rightarrow$$

$$\Rightarrow u(t, x) = \begin{cases} 1 & \text{if } x - t < 0 \\ 2 & \text{if } x - 2t > 0 \end{cases} = \begin{cases} 1 & \text{if } x < t \\ 2 & \text{if } x > 2t. \end{cases}$$

→ Thus, not uniquely define for $t < x < 2t$ ||.

Extra paper



Extra paper