Name Eduardo García-Juárez

Math 312 - Section 002 - Midterm 1
Thursday, February 14, 2019, @ 10:30 AM - 11:50 AM

No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________________

The use of calculators, computers and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. You may use both sides of one 8×11 cheat-sheet.

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Problem	Points	Your score
1	25	
2	15	
3	20	
4	20	
5	20	
Total	100	

Problem 1 [25 points]

Consider the matrix $A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 2 & 0 & -2 & 6 \end{bmatrix}$.

Part a. [5 points] Find an $A = LU$ decomposition (L lower triangular, U upper triangular).

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 2 & 0 & -2 & 6 \end{bmatrix} \xrightarrow{R3' = R3 - 2R1} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -2 & 0 & 2 \end{bmatrix} \xrightarrow{R3'' = R3 + 2R2} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

and

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$ (Check: $LU = A$).

The decomposition of the above matrix A into the row reduced echelon form $EA = R$ is given by

$$R = \begin{bmatrix} \boxed{1} & 0 & -1 & 3 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

Part b. [5 points] Find a basis $\mathcal{B}$ for the column space of A, $C(A)$.

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ (first two columns of A).

Part c. [5 points] Find a basis for the nullspace of A, $N(A)$.

$A\vec{x}=\vec{0} \Leftrightarrow R\vec{x}=\vec{0}$

$$\begin{bmatrix} \boxed{1} & 0 & -1 & 3 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_3=\alpha \\ x_4=\beta \end{array} \quad \begin{array}{l} x_1=\alpha-3\beta \\ x_2=\beta \end{array} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\uparrow \uparrow$ free

• Basis for $N(A)$ $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ (Check: $A\vec{x}=\vec{0}$)

Part d. [5 points] Find the complete solution to $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ and $A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

$A\vec{x}=\vec{b} \Leftrightarrow \underbrace{EA}_{R}\vec{x}=E\vec{b}$.

$$E = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

$$E\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \rightarrow \text{No solution. } (0=2!)$$

$$E\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Part e. [5 points] Find the coordinates of $\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ in the basis $\mathcal{B}$ you found in part b.

From part d), we see that $A\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ has as a solution the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. That is,

$$\left[\vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3 \mid \vec{a}_4\right]\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 1\cdot\vec{a}_1 + 1\cdot\vec{a}_2 \Rightarrow$$

$$\Rightarrow \vec{b}\big|_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Problem 2 [15 points]

Construct a matrix with the following properties or explain why it is not possible:

Part a. [5 points] A 3 by 3 matrix with $C(A) = \text{span}\left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix} \right\}$, $N(A) = \text{span}\left\{ \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.

$$A = \begin{bmatrix}1 & 2 & a\\2 & 1 & b\\1 & 1 & c\end{bmatrix} \Rightarrow C(A) \checkmark \qquad \begin{bmatrix}1 & 2 & a\\2 & 1 & b\\1 & 1 & c\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$A\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow N(A) \checkmark$$

So, $A = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\1 & 1 & 0\end{bmatrix}$ satisfies the properties.

Part b. [5 points] A matrix whose column space is spanned by $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}$, and whose nullspace is spanned by $\begin{bmatrix}1\\2\\3\\6\end{bmatrix}$. Impossible.

$\dim C(A) = 2$

$\dim N(A) = 1$

But A is $m \times n$ with $m = 3$, $n = 4$

Rank-Nullity theorem requires $\dim C(A) + \dim N(A) = n$.

Here, $2 + 1 \neq 4$!!

Part c. [5 points] A 2 by 2 matrix A such that the complete solution to the system $A\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$ is given by $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \alpha\begin{bmatrix}2\\1\end{bmatrix}$, ($\alpha \in \mathbb{R}$).

$$A = \begin{bmatrix}1 & a\\0 & 0\end{bmatrix} \Leftarrow \left\{ \begin{array}{l} \text{One free variable} \Rightarrow \dim N(A) = 1 \Rightarrow \\ \Rightarrow \dim C(A) = 2 - 1 = 1 \Rightarrow 1 \text{ pivot}. \end{array} \right.$$

We only need a. But from the complete solution, we know that $x_2 = \alpha \Rightarrow x_1 = 1 + 2x_2 \Rightarrow x_1 - 2x_2 = 1 \Rightarrow \underline{\underline{a = -2}}$

Problem 3 [20 points]

Let V be the set of points (w, x, y, z) in $\mathbb{R}^4$ such that $w+x+y+z = 0$ and $2w+x-y-z = 0$.

Part a. [5 points] Find a basis for V.

We notice that V is given by the vectors $\vec{x}$ that solve

$B\vec{x} = \vec{0}$, with $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \end{bmatrix}$. That is, $V = N(B)$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & -2 & -2 \\ 0 & \boxed{1} & 3 & 3 \end{bmatrix}$$

free

$x_3 = \alpha$

$x_4 = \beta$

$x_1 = 2\alpha + 2\beta$

$x_2 = -3\alpha - 3\beta$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \to \text{Basis for } V = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Part b. [5 points] Find a matrix with four rows and four columns that satisfies $C(A) = V$.

We have that $C(A) = V = \text{span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Let's choose these two vectors as columns of A.

$$A = \begin{bmatrix} 2 & 2 & & \\ -3 & -3 & & \\ 1 & 0 & & \\ 0 & 1 & & \end{bmatrix}$$

Now, we need to be careful. The other two columns have to be lin. dependent with the other two.

(Because otherwise $\dim C(A) > 2 \to C(A) \neq V$).

The easiest example: $A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ -3 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Part c. [5 points] Let $\vec{b} = \begin{bmatrix} 4 \\ -6 \\ -1 \\ 3 \end{bmatrix}$. Can we ensure that, for any matrix A satisfying part b), the system $A\vec{x} = \vec{b}$ has infinite solutions? Explain your answer.

It's easy to check that $\vec{b}$ satisfies both equations that define V. That is, $\vec{b} \in V$. But since $V = C(A)$, we conclude that $\vec{b} \in C(A)$.

We know that $A\vec{x} = \vec{b}$ has a solution $\Leftrightarrow \vec{b} \in C(A)$, so $A\vec{x} = \vec{b}$ has indeed a solution.

- But moreover, $\dim C(A) = \dim V = 2 \Rightarrow \dim N(A) = 2 \Rightarrow$
$\Rightarrow$ 2 free variables $\Rightarrow$ If there is a solution, there are infinitely many.

In summary, $A\vec{x} = \vec{b}$ has to have infinitely many solutions.

Part d. [5 points] Find the coordinates of $\vec{b}$ in the basis $\mathcal{V} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$\vec{b} = \alpha_1 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \vec{b}\,|_{\mathcal{V}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}.$$

Let's find α_1, α_2: $\begin{bmatrix} 2 & 2 \\ -3 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -1 \\ 3 \end{bmatrix} \sim$

$$\sim \left[\begin{array}{cc|c} 2 & 2 & 4 \\ -3 & -3 & -6 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} \alpha_1 = -1 \\ \alpha_2 = 3 \end{matrix} \Bigg\} \Rightarrow \vec{b}\,|_{\mathcal{V}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

(Check it).

- <u>Remark</u>: You could use part d.) to answer the previous questions too.

Problem 4 [20 points]

Part a. [10 points] Given the following bases for for the space of polynomials of degree at most 2, $\mathcal{P}_2$, find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

$$\mathcal{B} = \{x-2, x^2+2x+3, 3x^2\},$$
$$\mathcal{C} = \{x^2+x+1, x+1, 1\}.$$

$$P_{C\leftarrow B} = M_C^{-1} M_B \text{ where } M_C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_B = \begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix},$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -2 & 3 & 0 \\ 1 & 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -2 & 3 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & -2 & 3 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & -3 \end{array}\right] \to P_{C\leftarrow B} = \begin{array}{c} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & -3 \\ -3 & 1 & 0 \end{bmatrix} \\ [\vec{b}_1]_C \quad [\vec{b}_2]_C \quad [\vec{b}_3]_C \end{array}$$

(Easy to check!)

(where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$).

Part b. [5 points] Write the polynomial $1+3x-x^2$ in terms of the basis $\mathcal{B}$.

$$1+3x-x^2 = \alpha_1(x-2) + \alpha_2(x^2+2x+3) + \alpha_3(3x^2) \Rightarrow$$

$$\Rightarrow \begin{bmatrix} -2 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 7 & 0 & 7 \\ 0 & 1 & 3 & -1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & -2 \end{array}\right] \Rightarrow \left.\begin{array}{l} \alpha_1 = 1 \\ \alpha_2 = 1 \\ \alpha_3 = -2/3 \end{array}\right\} \Rightarrow$$

$$\Rightarrow 1+3x-x^2 = (x-2) + (x^2+2x+3) - \frac{2}{3}(3x^2) /\!/.$$

Part c. [5 points] Write the polynomial $1 + 3x - x^2$ in terms of the basis $\mathcal{C}$.

$$p(x) = 1 + 3x - x^2 \Rightarrow p|_B = \begin{bmatrix} 1 \\ 1 \\ -2/3 \end{bmatrix} \Rightarrow$$

$$\Rightarrow p|_C = P_{C \leftarrow B}\, p|_B = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & -3 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}$$

That is,

$$1 + 3x - x^2 = -(x^2 + x + 1) + 4 \cdot (x+1) - 2 \cdot 1 \quad //$$

Problem 5 [20 points]

In each of the following cases, clearly mark the statement as **true** or **false**. Please also explain your answers in order to receive credit for this problem.

1. Let A be a 3 by 3 matrix. If the left-nullspace of A is given by the vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying $x_1 + x_2 + x_3 = 0$, then A cannot be invertible.

True: $x_1 + x_2 + x_3 = 0$ is a plane in $\mathbb{R}^3$, thus has dimension 2. That is, $\dim N(A^T) = 2 \Rightarrow \dim C(A^T) = 3-2 = 1$. Therefore, $\dim C(A) = 1 < 3 \Rightarrow A$ not invertible.

2. If A and B are matrices with rank 1, then so is $(A+B)/2$.

False: Counterexample,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

3. If P_{12} is the matrix that changes row 1 with row 2, then $P_{12}^2 = I$ (I denotes the identity matrix).

True: $P_{12}^{-1} = P_{12} \Rightarrow P_{12}^2 = P_{12} P_{12}^{-1} = I$.

(also valid a simple word explanation).

4. If a 4x3 matrix has 3 pivots, then $A\vec{x} = \vec{b}$ always has at least a solution.

False:

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$ has no solution at all.

5. The set of 3 by 3 matrices with the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ in their column space is a vector space.

False: The zero vector, i.e., $A = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ is not in the set.

(Better: $A = \begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}$ is in the set, and $-A$ too ($\Rightarrow$ But $A + (-A)$ is not.))

6. The set of points satisfying both $y = 3x + z$ and $z + x = 0$ is a subspace of $\mathbb{R}^3$.

True: Notice that the set is the nullspace of

$$\begin{bmatrix}3 & -1 & 1\\1 & 0 & 1\end{bmatrix}.$$

7. The matrices A and A^T always have the same number of pivots.

True: $\dim C(A) = \operatorname{rank}(A) = \dim C(A^T)$.

8. If A is a change of basis matrix from a basis $\{\vec{v}_1, \vec{v}_2\}$ to a basis $\{\vec{u}_1, \vec{u}_2\}$, and B is a change of basis matrix from the basis $\{\vec{u}_1, \vec{u}_2\}$ to a basis $\{\vec{w}_1, \vec{w}_2\}$, then AB is a change of basis matrix from $\{\vec{v}_1, \vec{v}_2\}$ to $\{\vec{w}_1, \vec{w}_2\}$.

False:

$$\left.\begin{aligned}\vec{b}|_u &= A\,\vec{b}|_v\\ \vec{b}|_w &= B\,\vec{b}|_u\end{aligned}\right\} \Rightarrow \vec{b}|_w = BA\,\vec{b}|_v.$$

In general, $BA \neq AB$.

Extra space for work:

Extra space for work:

Extra space for work:

Extra space for work: