Practice exam: Solutions

Problem 1

$$\left.\begin{array}{l} u_t - u_{xx} + u = 0 \\ u(x,0) = e^{2x} \end{array}\right\} \left(\begin{array}{c} -\infty < x < \infty \\ t > 0 \end{array}\right)$$

Sol:

Let $u(x,t) = v(x,t)e^{-t} \Rightarrow \left| \begin{array}{l} v_t(x,t)e^{-t} - v(x,t)e^{-t} = u_t(x,t) \\ v_{xx}(x,t)e^{-t} = u_{xx}(x,t) \end{array} \right\} \Rightarrow$

$$\Rightarrow v_t(x,t)e^{-t} - \cancel{v(x,t)}e^{-t} - v_{xx}(x,t)e^{-t} + \cancel{v(x,t)}e^{-t} = 0 \xrightarrow{(e^{-t} \neq 0 \ \forall t \geq 0)}$$

$$\Rightarrow \left.\begin{array}{l} v_t - v_{xx} = 0 \\ v(x,0) = e^{2x} \end{array}\right\}$$ Homogeneous heat equation on the whole line with initial data e^{2x}.

$$\hookrightarrow \boxed{v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} e^{2y}\, dy}$$

Finally we try to simplify the formula:

Let's complete the square in "y":

$$-\frac{(x-y)^2}{4t} + 2y = -\frac{(x-y)^2 - 8ty}{4t} = -\frac{x^2+y^2-2y(x+4t)}{4t} =$$

$$= -\frac{(y-(x+4t))^2 - (4t)^2 - 8xt}{4t} = -\frac{(y-(x+4t))^2}{4t} + 4t + 2x.$$

Thus,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(y-(x+4t))^2}{4t}}\, e^{4t+2x}\,dy = \left\{ \begin{array}{l} z = \frac{y-(x+4t)}{\sqrt{4t}} \\ dz = \frac{dy}{\sqrt{4t}} \end{array} \right\}$$

$$= \frac{e^{4t}e^{2x}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-z^2}\sqrt{4t}\,dz = e^{4t}e^{2x}\frac{1}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-z^2}\,dz}_{=\sqrt{\pi}} = e^{4t}e^{2x}.$$

Therefore,

$$\left\| u(x,t) = e^{3t}e^{2x} \right\|$$

Check: $u(x,0) = e^{2x}$ ✓

$$\left.\begin{array}{l} u_t(x,t) = 3e^{3t}e^{2x} \\ u_{xx}(x,t) = 4e^{3t}e^{2x} \end{array}\right\} \Rightarrow u_t - u_{xx} + u = 0 \ ✓$$

Problem 2

Done in homework:

1) Inhomogeneous BC → modify the unknown so the BC becomes homogeneous.

Here,

let $u(x,t) = v(x,t) - h(t) \Rightarrow u(0,t) = v(0,t) - h(t) = 0$.

Then, the problem rewrites in $u(x,t)$ as

$$\left.\begin{array}{l} u_t(x,t) + h'(t) - k\,u_{xx}(x,t) = f(x,t) \\ u(0,t) = 0 \\ u(x,0) = v(x,0) - h(0) = \phi(x) - h(0) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} u_t(x,t) - k\,u_{xx}(x,t) = f(x,t) - h'(t) \\ u(0,t) = 0 \\ u(x,0) = \phi(x) - h(0) \end{array}\right\} \begin{bmatrix} 0 < x < \infty \\ t > 0 \end{bmatrix}$$

2) Half-line with Dirichlet BC → odd extension of initial data and force.

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Problem 3

$$\left.\begin{aligned} &u_{tt} = 2u_{xx} + u \\ &u_x(0,t) = u_x(\pi,t) = 0 \\ &u(x,0) = 1 + 3\cos(x) \\ &u_t(x,0) = 1 + 7\cos(2x) \end{aligned}\right\} \quad \begin{aligned} &x \in (0,\pi) \\ &t > 0 \end{aligned}$$

Solution:

Separation of variables: $u(x,t) = X(x)T(t)$

Then,

$$X(x)T''(t) = 2X''(x)T(t) + X(x)T(t) \rightarrow$$

$$\rightarrow \frac{T''(t)}{2T(t)} - \frac{1}{2} = \frac{X''(x)}{X(x)} = \text{constant} = -\lambda$$

1) BVP

$$\left.\begin{aligned} &X''(x) + \lambda X(x) = 0 \\ &X'(0) = X'(\pi) = 0 \end{aligned}\right\}$$

Neumann BC $\rightarrow$ Only nonnegative eigenvalue ($\lambda \geq 0$).

$$\begin{pmatrix} u_x(0,t) = X'(0)T(t) = 0 \;\; \forall t > 0 \\ u_x(\pi,t) = X'(\pi)\,T(t) = 0 \;\; \forall t > 0 \end{pmatrix}$$

- $\lambda = 0 \to X(x) = Ax + B$

$$\left.\begin{array}{l} X(x) = Ax+B \\ X'(0) = A = X'(\pi) = 0 \end{array}\right\} \Rightarrow X_0(x) = 1 \text{ is an eigenfunction.}$$

- $\lambda = \beta^2 > 0 \to X(x) = A\cos(\beta x) + B\sin(\beta x)$

$X'(0) = B\beta = 0 \Rightarrow B = 0$

$X'(\pi) = -A\beta\sin(\beta\pi) = 0$ $\nearrow A = 0$ (trivial) $\searrow \beta\pi = n\pi, \ n \in \mathbb{Z}$.

In summary,

$$\| \lambda_n = n^2 \text{ with } X_n(x) = \cos(nx) \|, \ n \geq 0.$$

2) <u>I.C</u>

$u(x,t) = \sum_{n=0}^{\infty} T_n(t)\cos(nx)$, where $T_n(t)$ is given by:

$T_n''(t) + (2\lambda_n - 1)T_n(t) = 0$

$\hookrightarrow T_n''(t) + (2n^2 - 1)T_n(t) = 0$

2.1) $n = 0 \to T_0(t) = a_0 e^{t} + b_0 e^{-t}$

2.2) $n \geq 1 \to T_n(t) = a_n \cos(\sqrt{2n^2-1}\, t) + b_n \sin(\sqrt{2n^2-1}\, t)$

Finally,

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\left(\sqrt{2n^2-1}\,t\right)+b_n\sin\left(\sqrt{2n^2-1}\,t\right)\right)\cos(nx)+$$

$$+\left(a_0e^{t}+b_0e^{-t}\right),$$

where the coefficients are determined by

$$\left.\begin{aligned} u(x,0)&=1+3\cos(x)=a_0+b_0+\sum_{n=1}^{\infty}a_n\cos(nx)\\ u_t(x,0)&=1+7\cos(2x)=a_0-b_0+\sum_{n=1}^{\infty}b_n\sqrt{2n^2-1}\cos(nx)\end{aligned}\right\}\rightarrow$$

$$\rightarrow\quad a_0=1,\quad a_1=3,\quad a_n=0 \text{ for } n\neq 1,3$$

$$b_0=0,\quad b_2\sqrt{7}=7\rightarrow b_2=\sqrt{7}$$

That is,

$$\boxed{u(x,t)=e^{t}+3\cos(t)\cos(x)+\sqrt{7}\sin(\sqrt{7}\,t)\cos(2x)}$$

Check: $u(x,0)=1+3\cos(x)$ ✓

$u_t(x,0)=1+7\cos(2x)$ ✓

$u_x(0,t)=0=u_x(\pi,t)$ ✓

$u_{tt}=2u_{xx}+u$ ✓

Problem 1

a) Let X_1, X_2 be two eigenfunctions. If we prove that the BC are symmetric, then the eigenvalues are real:

$$X_1'(x)X_2(x) - X_1(x)X_2'(x)\Big]_0^1 =$$

$$= X_1'(1)X_2(1) - X_1(1)X_2'(1) - X_1'(0)\underbrace{\cancel{X_2(0)}}_{=0} + \underbrace{\cancel{X_1(0)}}_{=0}X_2'(0) =$$

$$= 3X_1(1)X_2(1) - X_1(1)\,3X_2(1) = 0 \Rightarrow \text{symmetric BC.}$$

b) Clearly there are infinite positive eigenvalues:

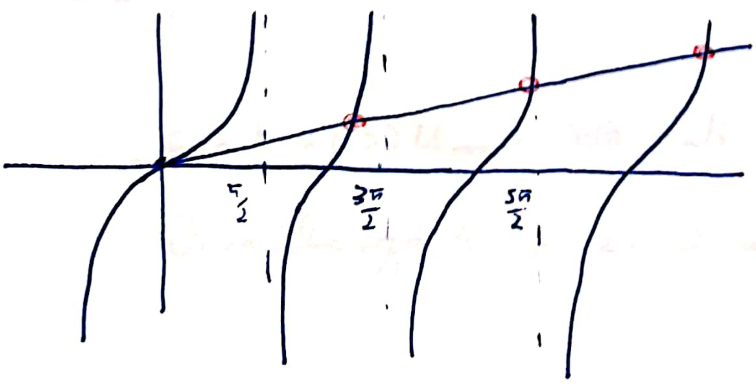
Let $\lambda = \beta^2 > 0$: $X'' + \lambda X = 0 \Rightarrow X(x) = A\cos(\beta x) + B\sin(\beta x)$.

Impose BC: $X(0) = A = 0$

$$3X(1) - X'(1) = 0 = 3B\sin(\beta) - B\beta\cos(\beta) \Rightarrow$$

$\Rightarrow \tan(\beta) = \beta/3 \rightsquigarrow$ there is a solution on each interval

$\left((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2}\right)$, $n \geq 1$.



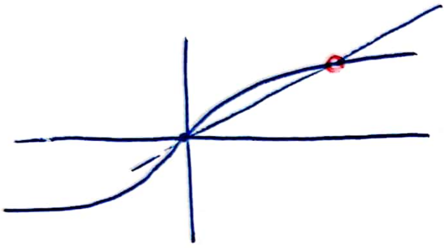
c) Let $\lambda = -\beta^2 < 0$:

$X(x) = Ae^{\beta x} + Be^{-\beta x}$,

$X(0) = A + B = 0 \rightarrow X(x) = A(e^{\beta x} - e^{-\beta x}) = A\,2\sinh(\beta x)$.

$3X(1) - X'(1) = 6A\sinh(\beta) - 2A\beta\cosh(\beta) = 0 \Leftrightarrow$

$\Leftrightarrow \tanh(\beta) = \frac{\beta}{3} \rightarrow$ Only one value of $\beta \neq 0$ solves this equation



Problem 5

a) $\displaystyle\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right)$, where

$$A_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^1 \sin\left(\frac{n\pi x}{2}\right)dx = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big]_0^1 =$$

$$= \frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right) = \begin{cases} \dfrac{2}{(2k-1)\pi}, & n = 2k-1 \\[2mm] \dfrac{1-(-1)^k}{k\pi}, & n = 2k \end{cases} \qquad (k \geq 1).$$

$$\text{So } A_n = \begin{cases} \dfrac{2}{n\pi} & \text{if } \begin{array}{l} n = 2k-1 \\ n = 2k,\ k \text{ odd} \end{array} \\[4mm] 0 & n = 2k,\ k \text{ even}. \end{cases}$$

So the series is

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \frac{2}{\pi}\left(\sin\left(\frac{\pi x}{2}\right) + \sin(\pi x) + \frac{1}{3}\sin\left(\frac{3\pi x}{2}\right) + \right.$$

$$\left. + \frac{1}{5}\sin\left(\frac{5\pi x}{2}\right) + \frac{1}{3}\sin(3\pi x) + \ldots\right).$$

b) $f(x)$ is piecewise continuous and $f'(x)$ piecewise continuous on $[0, 2]$,

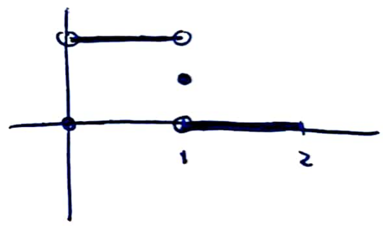
thus the series converges pointwise to $\frac{1}{2}(f(x^+) + f(x^-))$ for all $x \in (0,2)$:

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \begin{cases} 1 & 0 < x < 1 \\ 1/2 & x = 1 \\ 0 & 1 < x < 2 . \end{cases}$$

At $x = 0$, obviously the sum is 0, and same reasoning for $x = 2$.

Thus,

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{2}\right) = \begin{cases} 0 & x = 0 \\ 1 & 0 < x < 1 \\ 1/2 & x = 1 \\ 0 & 1 < x \leq 2 \end{cases}$$



c) No: each term is continuous, so if the series converges uniformly then the limit should be continuous. $f(x)$ is not.

[The theorems or Weierstrass test don't apply to this example].

d) Yes, because $f(x) \in L^2$ on $(0,2)$, i.e., $\int_0^2 |f(x)|^2 \, dx = 1 < +\infty$ //.