Practice Exam: Solutions

Problem 1

$y = a + b\,2^t, \qquad t = 0, 1, 2$

$\qquad\qquad\qquad\quad y = 6, 4, 0$

a) 1)

$$\left.\begin{aligned} 6 &= a + b \\ 4 &= a + 2b \\ 0 &= a + 4b \end{aligned}\right\} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}}_{\vec{b}}$$

2) The best fit in the least-squares sense is given by the solution to the normal equations:

$$A^T A \vec{x} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

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Remark: In this problem, you can realize that $a = 8$, $b = -2$ is an exact solution! Indeed, the equations in part a) are solvable.

[This is almost never the case in least-squares problems, so I solved the problem as if I didn't realize it].

If you do notice from start, then part 2) is unnecessary (since we know that if a system has a solution, then the least-squares one is that same one).

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b)

1) $\vec{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ eigenvector of A $\rightarrow A\vec{v} = \lambda \vec{v}$ (*).

$$A^T \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \rightarrow \begin{bmatrix}1\\0\\0\end{bmatrix} \in N(A^T)$$

- If the corresponding eigenvalue of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is not zero ($\lambda \neq 0$), then $\begin{bmatrix}1\\1\\1\end{bmatrix} \in C(A)$. but then

$$\left.\begin{array}{l}\begin{bmatrix}1\\1\\1\end{bmatrix} \in C(A)\\ \begin{bmatrix}1\\0\\0\end{bmatrix} \in N(A^T)\end{array}\right\}$$ contradiction since $\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = 1 \neq 0$ $(C(A) \perp N(A^T))$.

- If $\begin{bmatrix}1\\1\\1\end{bmatrix}$ correspond to $\lambda = 0$, then $\begin{bmatrix}1\\1\\1\end{bmatrix} \in N(A)$.

So we need A such that

$$\left.\begin{array}{l}\begin{bmatrix}1\\0\\0\end{bmatrix} \in N(A^T) \rightarrow \text{First row of A are zeros}\\ \begin{bmatrix}1\\1\\1\end{bmatrix} \in N(A) \rightarrow \text{columns of A add up to zero}\end{array}\right\}$$

so for example $A = \begin{bmatrix}0 & 0 & 0\\1 & -1 & 0\\1 & -1 & 0\end{bmatrix}$ //.

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2) Rows add up to a row of zeros $\Rightarrow \begin{bmatrix}1\\1\\1\end{bmatrix} \in N(A^T)$

Columns add up to a column of 1's $\Rightarrow \begin{bmatrix}1\\1\\1\end{bmatrix} \in C(A)$ $\Big\} \Rightarrow$

$\Rightarrow$ But this is not possible since $N(A^T)$ and $C(A)$ are orthogonal complements.

3) Possible: example $\rightarrow$ $A = \begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}$

4) Possible: $A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

↳ Obviously, $A = \underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{S} \underbrace{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}}_{\text{diagonal}} \underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^{-1}}_{S^{-1}}$ and $\lambda = 0$ is an eigenvalue//.

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Problem 2

a)

- $A = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$ is not : $\lambda_1 = 3 = \lambda_2$, but

$A - 3I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow \dim N(A-3I) = 1 < 2$ //.

- $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 5 \\ 1 & 3 & 5 \end{bmatrix}$ is diagonalizable :

its eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 8$ (three distinct eigenvalues).

Remark: $\lambda_1 = 0$ since B is singular, $\lambda_2 = 1$ because clearly $B - I$ is singular, and finally

$\lambda_3 = \text{trace}(B) - 0 - 1 = 9 - 1 = 8$ //.

- $C = \begin{bmatrix} -7 & 13 \\ 13 & 1 \end{bmatrix}$

Eigenvalues: $\det(C - \lambda I) = (-7-\lambda)(1-\lambda) - 13^2 =$

$= \lambda^2 + 6\lambda - 7 - 169 = \lambda^2 + 6\lambda - 176 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda = \dfrac{-6 \pm \sqrt{36 + 4 \cdot 176}}{2} \Rightarrow$ two distinct eigenvalues $\Rightarrow$

$\Rightarrow$ diagonalizable //.

Remark: We'll see that all symmetric matrices are diagonalizables //.

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b) Let's choose B (it has integer eigenvalues).

$$N(B): \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 5 \\ 1 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{3} & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$$

$$N(B-I): \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 3 & 4 \end{bmatrix}}_{B-I} \to \begin{bmatrix} 0 & 0 & 0 \\ \boxed{1} & 2 & 5 \\ 0 & \boxed{1} & -1 \end{bmatrix} \to \vec{v}_2 = \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

(do more ~~st~~ steps if needed)

$$N(B-8I): \underbrace{\begin{bmatrix} -7 & 0 & 0 \\ 1 & -5 & 5 \\ 1 & 3 & -3 \end{bmatrix}}_{B-8I} \to \begin{bmatrix} 0 & 0 & 0 \\ 1 & -5 & 5 \\ 0 & 8 & -8 \end{bmatrix} \to v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

So

$$B = \begin{bmatrix} 0 & -7 & 0 \\ -5 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 0 & -7 & 0 \\ -5 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}^{-1}$$

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c)

$$A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-2-\lambda)+4 = \lambda^2 - 4 + 4 = \lambda^2 = 0 \rightarrow \lambda_1 = \lambda_2 = 0 .$$

But $\dim N(A) = 1 < 2$ //

$$\left(\uparrow \quad A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ \boxed{1} & -2 \end{bmatrix} \right)$$

d)

$$A^2 = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(We knew this by Cayley-Hamilton theorem)

$$e^{At} = I + At + \frac{1}{2!}\cancel{A^2 t^2} + \frac{1}{3!}\cancel{A^3 t^3} + \dots =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & -4t \\ t & -2t \end{bmatrix} = \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} // .$$

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e) $\vec{x}(t) = e^{At}\,\vec{x}(0) \Rightarrow$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+6t-4t \\ 3t+1-2t \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} x(t) = 3+2t \\ y(t) = t+1 \end{cases}$$ (check it ✓).

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Problem 3

$$V \equiv -w + x - 2z = 0 \rightsquigarrow \begin{bmatrix} -1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = 0 .$$

three "free variables" & $\dim V = 3$.

a)

$$\text{Basis for } V = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{a}_2}, \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{\vec{a}_3}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{a}_1} \right\}$$

(reorder for convenience)

b)

$$\vec{v}_1 = \vec{a}_1$$

$$\vec{v}_2 = \vec{a}_2 - \underbrace{\left(\frac{\vec{v}_1 \cdot \vec{a}_2}{\|\vec{v}_1\|^2} \right)}_{=0} \vec{v}_1 = \vec{a}_2$$

$$\vec{v}_3 = \vec{a}_3 - \underbrace{\left(\frac{\vec{v}_1 \cdot \vec{a}_3}{\|\vec{v}_1\|^2} \right)}_{=0} \vec{v}_1 - \left(\frac{\vec{v}_2 \cdot \vec{a}_3}{\|\vec{v}_2\|^2} \right) \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

→ check: $\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = 0$.

• Normalize: $\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$, $\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$, $\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|}$

So orthonormal basis for V $= \{ \vec{q}_1, \vec{q}_2, \vec{q}_3 \} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \end{bmatrix} \right\}$

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c) That is $P_V \vec{b}$, with $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$.

Two easy ways:

(1) $P_V \vec{b} = (\vec{q}_1 \cdot \vec{b})\, \vec{q}_1 + (\vec{q}_2 \cdot \vec{b})\, \vec{q}_2 + (\vec{q}_3 \cdot \vec{b})\, \vec{q}_3$.

or

(2) $P_V \vec{b} = \vec{b} - P_{V^\perp} \vec{b}$, since a basis for $V^\perp$ is simply $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \end{bmatrix} \right\} \equiv \{ \vec{u} \}$

[Remark: • Do not use the original basis $\{\vec{a}_2, \vec{a}_3, \vec{a}_1\}$ (you can, but it is silly: longer and easier to make mistakes)].

Method (2) has the advantage of being faster and it does not rely on your previous computations.

$$P_V \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} - \left(\frac{\vec{u} \cdot \vec{b}}{\|\vec{u}\|^2} \right) \vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} - \frac{-2}{6} \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2 \\ -2/3 \end{bmatrix}$$

(easy check: $P_V \vec{b} \in V$)
(i.e., satisfies the equation)

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Problem 4 [20 points]

Let $V = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 0\}$. Consider the following linear transformation T: projection of vectors in $\mathbb{R}^3$ onto V.

Part a. If P is the usual 3 by 3 projection matrix (i.e., the matrix of the linear transformation T using the standard basis), find three eigenvalues and three independent eigenvectors of P. (Hint: No need to compute P).

Solution: We recall from class that the null space $N(P) = P^{\perp}$. Since P has a nonzero nullspace and the nullspace is the eigenspace of $\lambda = 0$, we know 0 is an eigenvalue, and it has a basis which is any basis for $P^{\perp}$.

Since $P = N(\begin{bmatrix} 1 & 2 & 1 \end{bmatrix})$, we see that $P^{\perp} = C(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix})$, so $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace of 0.

We also recall that for any vector $\vec{v}$ in V, $P\vec{v} = \vec{v}$ (this is because the closest point in V to $\vec{v}$ is $\vec{v}$ itself!), so V is in the eigenspace of $\lambda = 1$, but V is two dimensional, so it must be the entire eigenspace of 1, so 0 and 1 are the only eigenvalues, and a basis for the eigenspace of 1 is any basis of V.

The basis we choose for the eigenspace of 1 is $\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$.

Part c. Find an orthonormal basis $\mathcal{V}$ for V.

Solution. Applying Gram–Schmidt to the above basis, we first get an orthogonal basis

$$\vec{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{-2}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Normalizing, we get $\vec{q}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{q}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$

Part d. Find the matrix of the linear transformation T when the input basis is $\mathcal{U}$,

$$\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \},$$

and the output basis is $\mathcal{V}$.

Solution. We calculate the projection of each of the basis vectors u onto P, using the nice formula for projection onto space using an orthonormal basis for that space.

$$proj_V \vec{u}_1 = (\vec{u}_1 \cdot \vec{q}_1)\vec{q}_1 + (\vec{u}_1 \cdot \vec{q}_2)\vec{q}_2$$

the coefficients of this form the first column of the matrix of the linear transformation. So the whole matrix of the linear transformation is

$$\begin{bmatrix} \vec{u}_1 \cdot \vec{q}_1 & \vec{u}_2 \cdot \vec{q}_1 & \vec{u}_3 \cdot \vec{q}_1 \\ \vec{u}_1 \cdot \vec{q}_2 & \vec{u}_2 \cdot \vec{q}_2 & \vec{u}_3 \cdot \vec{q}_2 \end{bmatrix}$$

(Final step: compute these dot products and enter them into the matrix)

Problem 5 [20 points]

In each of the following cases, clearly mark the statement as **true** or **false**. Please also explain your answers in order to receive credit for this problem.

1. If A is a 3x3 matrix with determinant 1, then $2A$ has determinant 6. **Solution**: False; counterexample: $A = Id$ has determinant 1, but

$$2A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

has determinant 8.

2. The matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$ has an eigenvalue equal to 9. **Solution**: Call this matrix A. So we want to check if $\det A - 9I$ is zero or not. If it is zero, then 9 is an eigenvalue. If not, 9 is not. You can row reduce the matrix $\det A - 9I$ to a matrix with a zero row, so the determinant is zero and TRUE, 9 is an eigenvalue.

3. An square matrix with orthonormal columns always has orthonormal rows. **Solution**: True: since the columns are orthonormal, $A^T A = Id$ but since A is square, this means $A^T = A^{-1}$ so $AA^T = Id$. But this means the columns of A^T are orthonormal. So the rows of A are orthonormal.

4. Let A be an n by n matrix. If n is odd and A is skew-symmetric (i.e., $A^T = -A$), then A is not invertible. **Solution**: True: factoring the -1 out of each row, we see $\det(A^T) = \det(-A) = (-1)^n \det(A) = (-1)^n \det(A^T)$ (we recall the determinant of the transpose is equal to determinant of original matrix). Since n is odd, this means $\det(A^T) = -\det(A^T)$, so the determinant must be zero so A is not invertible.

5. A 2 by 2 real matrix that rotates every vector 90º cannot have any real eigenvalues. **Solution**: True: rotation by 90 degrees does not send any vector to 0, so the matrix cannot have any eigenvectors with eigenvalue 0. And A cannot have eigenvectors with eigenvalue 1, since those would be fixed points $Ax = x$, and rotation by 90 degrees does not send any points to itself; nor can it have eigenvalue -1 since sending x to -x is not rotation by 90 degrees. Also, A cannot have real eigenvectors with eigenvalue λ for other values of λ, since that means rotation would scale the length of the vectors by $|\lambda|$, but rotation doesn't change the lengths.