Name ______________________________

Math 312 - Section 002 - Final Exam (Practice exam)

Thursday May 9, 2019, @ 12:00 - 2:00 PM

No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity. I certify that all of the work on this test is my own.

Signature: ______________________

Do not write on the back side of any page. The use of calculators, computers and similar devices is neither necessary nor permitted during this exam. Correct answers without proper justification will not receive full credit. Clearly highlight your answers and the steps taken to arrive at them: illegible work will not be graded. You may use both sides of one 8 × 11 cheat-sheet.

OFFICIAL USE ONLY:

Problem	Points	Your score
1	10	
2	10	
3	10	
4	10	
5	10	
6	15	
7	15	
8	20	
Total	100	

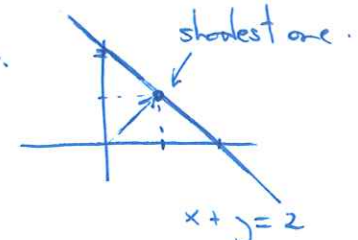
Problem 1 [10 points]

Part a. Find the solution of shortest length (distance from the origin) to the following system $A\vec{x} = \vec{b}$:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

The complete solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $(\alpha \in \mathbb{R})$.

It is clear that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in C(A^T)$, so it is the shortest length solution. Graphically:



<u>Remark</u>: You could use projection onto $C(A^T)$ or pseudoinverse, but not necessary here.

Part b. Find the coefficients for the model below that best fit the data $x = 0, -\pi/2, \pi$, $y = 0, 1, 0$ in the least squares sense:

$$y = a + b\cos x.$$

$$\left.\begin{aligned} 0 &= a + b \\ 1 &= a \\ 0 &= a - b \end{aligned}\right\} \rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}}$$

$$A^T A \vec{x} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad A^T \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \left| \begin{aligned} a &= 1/3 \\ b &= 0 \end{aligned} \right.$$

Problem 2 [10 points]

Part a. Which of the following matrices are diagonalizable over the complex numbers?

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -3 \\ -1 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{A}$: $\lambda_1 = 0$ (A is singular)
$\lambda_2 = 2$ (trace$(A) = 2$)

So A is diagonalizable
(two distinct eigenvalues)

$\underline{B}$ $(1-\lambda)(-1-\lambda) - 3 = 0$
$-1 + \lambda^2 - 3 = 0$
$\lambda^2 = 4 \Rightarrow \lambda = \pm 2 \rightarrow$ distinct
So B is diagonalizable.

$\underline{C}$: No diagonalizable
$\lambda_1 = 1 = \lambda_2 = \lambda_3$ (triangular)

But

$\dim N(C - I) = \dim N\left(\begin{bmatrix} 0 & \boxed{2} & 2 \\ 0 & 0 & \boxed{2} \\ 0 & 0 & 0 \end{bmatrix}\right) = 2 < 3$.

Consider the linear differential system

$$\begin{aligned} x' &= x - 2y \\ y' &= 2x + y. \end{aligned}$$

Part b. For which matrix A can we rewrite this system as $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$?

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Part c. Find the solution $x(t)$, $y(t)$ for $x(0) = 1 = y(0)$.

$$\det(A-\lambda I) = (1-\lambda)^2 + 4 = 0 \Rightarrow \begin{array}{l} \lambda_1 = 1+2i \\ \lambda_2 = 1-2i \end{array}$$

$$A-\lambda_1 = \begin{bmatrix} -2i & -2 \\ 2 & 2i \end{bmatrix} \to \vec{u}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad A-\lambda_2 = \begin{bmatrix} 2i & -2 \\ 2 & -2i \end{bmatrix} \to \vec{u}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

$$\vec{x}(t) = e^{At}\vec{x}(0) = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} i e^{t}(\cos(2t) + i\sin(2t)) & -i e^{t}(\cos(2t) - i\sin(2t)) \\ e^{t}(\cos(2t) + i\sin(2t)) & e^{t}(\cos(2t) - i\sin(2t)) \end{bmatrix} \begin{bmatrix} 1+i \\ -1+i \end{bmatrix} \frac{1}{2i} =$$

$$= \begin{bmatrix} 2i e^{t}\cos(2t) - 2i\sin(2t)e^{t} \\ 2ie^{t}\sin(2t) + 2i\cos(2t)e^{t} \end{bmatrix} \frac{1}{2i} = \begin{bmatrix} e^{t}(\cos(2t) - \sin(2t)) \\ e^{t}(\sin(2t) + \cos(2t)) \end{bmatrix} \Rightarrow$$

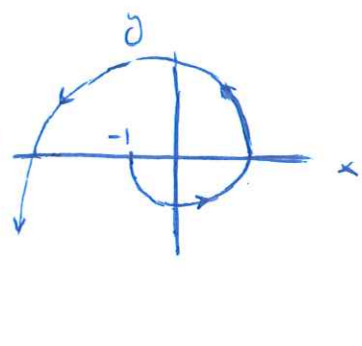
$$\Rightarrow \begin{array}{l} x(t) = e^{t}(\cos(2t) - \sin(2t)) \\ y(t) = e^{t}(\sin(2t) + \cos(2t)) \end{array}$$

Part d. Suppose $(x(t), y(t))$ is a solution with initial conditions $(x(0), y(0)) = (-1, 0)$. What is the limit of $(x(t), y(t))$ as $t \to \infty$.

Since the real part of the eigenvalues is positive, both $x(t)$, $y(t)$ increase exponentially fast, going to infinity.

• Indeed, $e^{At} = e^{t}\begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix}$

↳ $\vec{x}(t) = e^{At}\vec{x}(0) = -e^{t}\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} \to$



$\vec{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Problem 3 [10 points]

Let V be the subspace of vectors in $\mathbb{R}^4$ (with coordinates (x, y, z, w)) such that

$$x - w - 2z = 0.$$

Part a Find a basis for V.

$$\left\{ \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{a}_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{a}_2}, \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{a}_3} \right\}$$

Part b Apply Gram–Schmidt to find an orthonormal basis for V.

$\vec{v}_1 = \vec{a}_1$

$\vec{v}_2 = \vec{a}_2 - \cancel{\frac{\vec{a}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}} \vec{a}_1 = \vec{a}_2$

$\vec{v}_3 = \vec{a}_3 - \cancel{\frac{\vec{a}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}} \vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ $\left(\begin{array}{l} \vec{v}_3 \cdot \vec{v}_1 = 0 \\ \vec{v}_3 \cdot \vec{v}_2 = 0 \quad \checkmark \\ \vec{v}_1 \cdot \vec{v}_2 = 0 \end{array} \right)$

Finally, $\vec{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{q}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} / \sqrt{2}$, $\vec{q}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} / \sqrt{3}$, orthonormal basis for $V = \{ \vec{q}_1, \vec{q}_2, \vec{q}_3 \}$.

Part c Find the point on V closest to the point $(1, 1, 1, 1)$.

This is $P_V \vec{b}$ with $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. $V^\perp \rightsquigarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} \right\}$

$\vec{b} = P_V \vec{b} + P_{V^\perp} \vec{b} \rightsquigarrow P_V \vec{b} = \vec{b} - P_{V^\perp} \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-2}{6} \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1 \\ 1/3 \\ 2/3 \end{bmatrix}$

(Check: $\frac{4}{3} - \frac{2}{3} - \frac{2}{3} = 0 \checkmark$)

- Also:

$P_V \vec{b} = (\vec{b} \cdot \vec{q}_1) \vec{q}_1 + (\vec{b} \cdot \vec{q}_2) \vec{q}_2 + (\vec{b} \cdot \vec{q}_3) \vec{q}_3 = \ldots$

Problem 4 [10 points]

Part a. Let V be the vector space of polynomials of degree less or equal than 1, with basis $\{x, x-1\}$. Let U be the vector space of polynomials of degree less than or equal to 2, with basis $\{1, x+2, x^2+1\}$. Let $T: V \to U$ be the linear transformation which sends a polynomial p to the polynomial $(2x) \cdot p$ (for example, $T(x) = (2x)x = 2x^2$).

Find the matrix of the linear transformation T (multiplication by $2x$), using the bases for V and U as given above.

$$M_V^U = \begin{bmatrix} T(\vec{v}_1)|_U & T(\vec{v}_2)|_U \end{bmatrix}, \quad \begin{pmatrix} 3\times 2 \text{ since } \dim V = 2 \\ \dim U = 3 \end{pmatrix}$$

$$T(\vec{v}_1) = T(x) = 2x^2 \Rightarrow T(\vec{v}_1)|_U = (2(x^2+1)-2)|_U = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$T(\vec{v}_2) = T(x-1) = 2x(x-1) = 2x^2 - 2x = a \cdot 1 + b(x+2) + c(x^2+1) \Rightarrow$$

$$\Rightarrow c = 2,\ b = -2,\ a = 2 \Rightarrow T(\vec{v}_1)|_U = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

Thus $$M_V^U = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 2 & 2 \end{bmatrix}$$

Part b. Let V be the orthogonal complement of

$$Span\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)$$

in $\mathbb{R}^2$.

Part b.1. Find a basis for V.

$$\vec{x} \in V \text{ if } \vec{x} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = x_1 - 3x_2 = 0 \Rightarrow \text{Basis for } V = \left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$$

Part b.2. Find a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose kernel is V and which has the vector $\begin{bmatrix}2\\1\end{bmatrix}$ in its image.

$T(\vec{x}) = A\vec{x}$ with $N(A) = \left\{\begin{bmatrix}3\\1\end{bmatrix}\right\}$

$\begin{bmatrix}2\\1\end{bmatrix} \in C(A)$

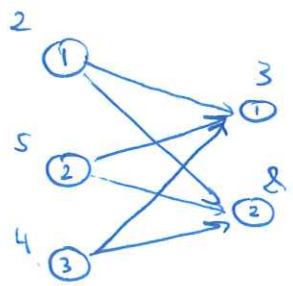
So for example,

$A = \begin{bmatrix}2 & a\\1 & b\end{bmatrix}$ with $\begin{bmatrix}2 & a\\1 & b\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \left| \begin{array}{l} a = -6 \\ b = -3 \end{array}\right.$

Problem 5 [10 points]

Part a. A firm produces goods at three different supply centers. The maximum supply produced at each supply center is, respectively, 2, 5, 4. The demand for the goods is spread out at two different demand centers. The demand at these centers is 3 and 8. The goal of the firm is to get goods from supply centers to demand centers at minimum cost, and all demands must be satisfied. Assume that the cost of shipping one unit of goods from supply center i to demand center j is c_{ij} $(i = 1, 2, 3,\ j = 1, 2)$ and that the cost is proportional to the quantity transported.

Write the problem as a Linear Programming problem in standard form.



Variables: x_{ij} = units of goods transported from center i to j.

Restrictions:

$$\left.\begin{aligned} x_{11} + x_{12} &\le 2 \\ x_{21} + x_{22} &\le 5 \\ x_{31} + x_{32} &\le 4 \end{aligned}\right\} \text{(maximum supply)}$$

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} &\ge 3 \\ x_{12} + x_{22} + x_{32} &\ge 8 \end{aligned}\right\} \text{(all demands must be satisfied)}$$

$x_{ij} \ge 0$ for $i = 1, 2, 3,\ j = 1, 2$.

Cost: $z = \sum_{i=1}^{3} \sum_{j=1}^{2} c_{ij} x_{ij} \to \min z$.

Standard form:

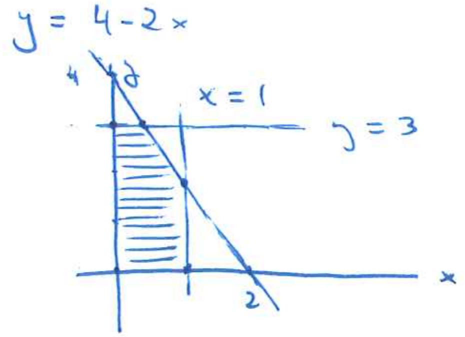
$$\left.\begin{aligned} -\max\ & -\sum_{i=1}^{3} \sum_{j=2}^{2} c_{ij} x_{ij} \\ \text{s.t. } & A\vec{x} \le \vec{b} \\ & \vec{x} \ge \vec{0} \end{aligned}\right\}$$

$$A = \begin{array}{c} \begin{array}{cccccc} x_{11} & x_{12} & x_{21} & x_{22} & x_{31} & x_{32} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{bmatrix} \end{array}$$

$$\vec{b} = \begin{bmatrix} 2 & 5 & 4 & -3 & -8 \end{bmatrix}^T.$$

Part b. Consider the linear programming problem where you have to maximize $f(x, y) = x + y$ for $x \geq 0$, $y \geq 0$, subject to the constraints $-2x + 4 \geq y$, $x \leq 1$ and $y \leq 3$.

Part b.1. Draw the feasible region of the problem.



Part b.2. Find the solution using geometrical methods (give the value of the maximum and the values of x, y where it is attained).

Corners: $(0,0)$, $(0,3)$, $(2,0)$, $(1,2)$ and $(\frac{1}{3}, 3)$

$\uparrow$

$\left.\begin{array}{l} y = 4-2x \\ y = 3 \end{array}\right\}$

$f(0,0) = 0$

$f(0,3) = 3$

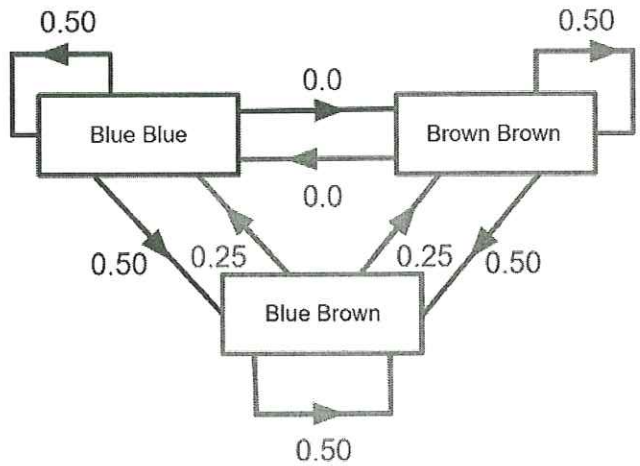
$f(2,0) = 2$

$f(1,2) = 3$

$f(\frac{1}{3}, 3) = 3 + \frac{1}{3}$ $\rightarrow$ maximum is $3 + \frac{1}{3}$ at $x = \frac{1}{3}$, $y = 3$ //.

Problem 6 [15 points]

Part a. Simple Mendelian genetics assumes that a trait, such as eye color, is determined by a combination of two genes, one inherited from each parent. The probability of a parent with a specific pair of genes having offspring with each gene combination can be illustrated by a transition diagram. Below is the transition diagram for eye color with Blue and Brown as the two possible genes.



Part a.1. Find the stochastic matrix M that models this Markov chain.

$$M = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} BB & BBr & BrBr \\ \end{array}$$

$$M = \left[\begin{array}{ccc} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{array}\right]\begin{array}{l} BB \\ BBr \\ BrBr \end{array}$$

Part a.2. What is probability that the grandchildren of a parent with the gene pair Brown Brown has the gene Blue Blue?

$$\vec{x}(2) = M^2\,\vec{x}(0) = MM\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = M\begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.5 \\ 0.3125 \end{bmatrix} \to \underline{\underline{12.5\%}}$$

(last column of M) (average of columns 2 and 3 of M)

Part a.3. In a population which currently only has Blue and Brown genes for eye color, which combination will be most common after a long period of time? Compute the fraction of people with that gene (the most common in the long term). And what if the population currently only has Brown Brown color? Explain why you can ensure that both answers have to be the same.

M does not have all positive entries, but we can easily check that $\lambda_1 = 0$ (1^{st} row is the 2^{nd} minus the 3^{rd}),

$\lambda_2 = 1$ (always for an stochastic matrix),

$\lambda_3 = 1/2$ (from the traces),

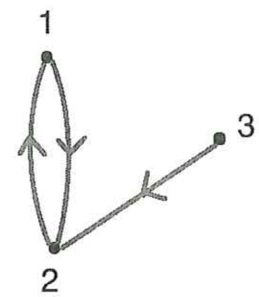
so $\lambda_2 = 1$ is non repeated and $|\lambda_3|, |\lambda_1| < 1$, thus there exist a steady state, and it is independent of the initial state.

It will therefore be given by the eigenvector of M with $\lambda = 1$:

$$M - I = \begin{bmatrix} -0.5 & 0.25 & 0 \\ 0.5 & -0.5 & 0.5 \\ 0 & 0.25 & -0.5 \end{bmatrix} \Rightarrow \vec{x}_\infty = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} / 4 = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$$

The most common ~~[illegible]~~ combination in the long term is Blue Brown, with 50% of people.

Part b.1. Find the transition matrix with damping $\alpha = 1/3$ of the following graph:



$$A = \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \rightsquigarrow A_\alpha = A(1-\alpha) + \frac{\alpha}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$

$$= \frac{2}{3}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1/3}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/9 & 7/9 & 1/9 \\ 7/9 & 1/9 & 7/9 \\ 1/9 & 1/9 & 1/9 \end{bmatrix}$$

Part b.2. Find the Page Rank (it suffices to list the pages in order of importance, from most to least), without damping (using the transition matrix with $\alpha = 0$).

$$A - I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \vec{x}_\infty = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$$ Pages 1,2 have the same PRank, then 3.

Problem 7 [15 points]

Part a Let

$$A = \begin{bmatrix} 3 & 0 & -4 \end{bmatrix}.$$

Find a basis for $N(A)$.

$$\begin{aligned} x_2 &= \alpha \\ x_3 &= \beta \\ 3x_1 &= 4\beta \end{aligned} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Part b Find the eigenvalues of $A^T A$ (with multiplicity).

$AA^T = 25 \Rightarrow$ Eigenvalues of A^TA are $\lambda_1 = 25$, $\lambda_2 = 0 = \lambda_3$.

Part c Find the SVD of A.

$\Sigma = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 1 \end{bmatrix}$ (only choice for a 1×1 orthogonal matrix).

$V \rightarrow$ First column can be found from $A = U\Sigma V^T$, the other two correspond to orthonormal basis of $N(A^TA) = N(A)$, so we can use Part a):

$$V = \begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix}^T$$

Part c Find the projection matrix onto the row space of A in terms of the matrices U, Σ, V in Part c.

$$P_{C(A^T)} = A^+A = V\Sigma^+U^TU\Sigma V^T = V\Sigma^+\Sigma V^T, \quad \Sigma^+ = \begin{bmatrix} 1/5 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 8 [20 points]

In each of the following cases, clearly mark the statement as **true** or **false**. Please also **explain your answers** in order to receive credit for this problem.

1. The collection of invertible 4x4 matrices form a subspace of the collection of all 4x4 matrices.

False: $A = I$, $B = -I$ are invertible, $A + B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ not invertible.

2. If matrices A and B satisfy $AB = I$, then A is invertible and $B = A^{-1}$.

False: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

3. A real matrix A has eigenvalues $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -2, \lambda_4 = 0$. Then, A cannot have all of its entries positive.

(Q. 117 T/F pdf)

4. If both $\underset{①}{A\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{b}_1}$ and $\underset{②}{A\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{b}_2}$ have solutions, then so does $\underset{③}{A\vec{x}} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{b}}$.

True:

$\vec{b} = \vec{b}_1 - 2\vec{b}_2$. Let $\vec{x}_1, \vec{x}_2$ be solutions of ① and ②.

Then, $\vec{x}_1 - 2\vec{x}_2$ solves ③.

5. If the standard form of an LP has $\vec{b} \geq 0$ and $\vec{c} \leq \vec{0}$, then the maximum is attained at the origin.

(Q. 134 T/F pdf)

6. If A is a 5 by 3 matrix, and $A = U\Sigma V^T$ is a singular value decomposition, and Σ has two nonzero entries, then the null space of AA^T has dimension 3.

(Q. 135 T/F pdf)

7. Let P be the matrix which projects vectors of $\mathbb{R}^3$ onto the plane $x + y + z = 0$. The eigenvalues of P are 0,1,1.

(Q. 75 T/F pdf)

8. If A is symmetric, then so is e^A.

(Q. 65 T/F pdf)

9. For any orthogonal matrix Q, all its singular values are 1.

(Q. 104 T/F pdf).

10. If a 4x3 matrix A has a RREF with three pivots, then $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$.

False: The columns of A don't span the whole $\mathbb{R}^4$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \to \text{no solution.}$$

11. If $\det(A^2) = 1$, then the eigenvalues of A are 1 or -1.

(Q. 62 T/F pdf)

12. If A is a stochastic matrix and $\vec{v}$ a vector, then the entries of $A\vec{v}$ sum up to the same quantity as the entries of $\vec{v}$.

(Q. 124 T/F pdf)

13. We know that the maximum positive eigenvalue of a real matrix A is $\lambda = 3$ with eigenvector $\vec{u} = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$. Then we can ensure that A have at least one non positive entry

(Q. 122 T/F pdf)

14. Every positive definite matrix is invertible.

True: The eigenvalues are all strictly positive, thus not zero, so the determinant $\neq 0$.

15. The matrix $A^T A$ is always positive semidefinite.

(Q 101 T/F pdf)

16. Given any basis of a vector space V, one can always find an orthonormal basis spanning V.

(Q. 33 T/F pdf)