MATH 425

LECTURE 18

Last days:

We were studying the "completeness" of Fourier series. More specifically,

given the eigenfunctions coming from the Sturm-Liouville

$$\left.\begin{array}{l} X'' + \lambda X = 0 \quad \text{in } (a,b) \\ \text{with (any) symmetric BC} \end{array}\right\} \quad [SL]$$

can we write any function on (a,b) as a series of these eigenfunctions $\{X_n\}$?

That is, given the family $\{X_n\}_{n\geq 1}$, in what sense can we write

$$[F] \quad f(x) = \sum_{n\geq 1} A_n X_n(x) \quad ?$$

We already know that:

1) The coefficients must be: $A_n = \dfrac{(f, X_n)}{(X_n, X_n)} = \dfrac{\int_a^b f(x) \overline{X_n(x)}\, dx}{\int_a^b |X(x)|^2\, dx}$. [C].

2) (Some) theorems for pointwise, uniform and L^2 convergence.
(i.e., conditions for [F] to hold and in what sense).

Today, we want to have an idea about those theorems.

IV] L^2 Theory

Recall the definitions:

Inner product: $(f,g) = \int_a^b f(x)\,\overline{g(x)}\,dx$.

L^2 norm: $\|f\|_{L^2} = (f,f)^{1/2} = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$

Distance in L^2 between f and g: $\|f-g\|_{L^2} = \left(\int_a^b |f(x)-g(x)|^2\right)^{1/2}$.

- In particular, two functions are said to be equal in L^2 if their distance is zero: $f = g$ in L^2 if $\|f-g\|_{L^2} = 0$.

- The previous day we mention "the completeness" of the Fourier series in L^2:

"**Theorem:** $f \in L^2 \Rightarrow f = \sum_{n=1}^{\infty} A_n X_n$ in L^2".

That is, if $\|f\|_{L^2} < +\infty$ (i.e. $f \in L^2$), then its Fourier series converges to f in L^2.

• Now, we are going to see that the partial sums of this Fourier series provides the best approximation to f (in L^2 sense).

More specifically,

Theorem: (Least-squares approximation)

Let $\{Y_n\}$ be <u>any</u> orthogonal set of functions.

Let $f \in L^2$ (i.e., $\|f\|_{L^2} < \infty$).

Let N be a fixed positive integer.

Then, the solution to

$$\min_{c_1, \dots, c_N} \left\| f - \sum_{n=1}^{N} c_n Y_n \right\|$$

is given by $c_n = \dfrac{(f, Y_n)}{(Y_n, Y_n)}$.

<u>Remark</u>: 1) $\{Y_n\}$ is any orthogonal set of functions. Not necessarily eigenfunctions.

2) The coefficients that gives the best approximation of f in the L^2 sense have the form of the Fourier coeff.

In other words, given a finite set of orthogonal functions $\{\Psi_n\}$, the linear combinations of these functions that best approximates f in L^2 sense is the "finite" Fourier series.

<u>Proof</u>: [Assume $f(x)$ and all the $\{\Psi_n\}$ are real valued].

The proof is direct: compute $\| f - \sum_{n=1}^{N} c_n \Psi_n \|$ and we will see the "best" c_n.

$$E_N = \| f - \sum_{n=1}^{N} c_n \Psi_n \|^2 = \int_a^b | f(x) - \sum_{n=1}^{N} c_n \Psi_n(x) |^2 dx =$$
(error)

$$= \int_a^b |f(x)|^2 dx - 2 \sum_{n=1}^{N} c_n \int_a^b f(x) \Psi_n(x)\, dx + \int_a^b \left(\sum_{n=1}^{N} c_n \Psi_n(x) \right) \left(\sum_{m=1}^{N} c_m \Psi_m(x) \right) dx =$$

$$= \|f\|_{L^2}^2 - 2 \sum_{n=1}^{N} c_n (f, \Psi_n) + \sum_{n=1}^{N} \sum_{m=1}^{N} c_n c_m \underbrace{\int_a^b \Psi_n(x) \Psi_m(x)\, dx}_{\neq 0 \text{ only if } n=m} =$$

$$= \|f\|_{L^2}^2 - 2 \sum_{n=1}^{N} c_n (f, \Psi_n) + \sum_{n=1}^{N} c_n^2 \|\Psi_n\|_{L^2}^2$$

Completing squares for c_n:

$$E_N = \|f\|_{L^2}^2 + \sum_{n=1}^{N} \|Y_n\|_{L^2}^2 \left(c_n^2 - 2c_n \frac{(f, Y_n)}{\|Y_n\|_{L^2}^2} \right) =$$

$$= \|f\|_{L^2}^2 + \underbrace{\sum_{n=1}^{N} \|Y_n\|_{L^2}^2 \left(c_n - \frac{(f, Y_n)}{\|Y_n\|_{L^2}^2} \right)^2}_{\geq 0} - \sum_{n=1}^{N} \frac{(f, Y_n)^2}{\|Y_n\|_{L^2}^2}$$

So, if we want to choose $c_1, c_2, \dots, c_N$ to minimize E_N, the best possible choice is the one that makes the middle term equal to zero $\Rightarrow$ $c_n = \dfrac{(f, Y_n)}{\|Y_n\|_{L^2}^2}$ //.

- Even more important, we have discovered Bessel's inequality:

Since $E_N = \|f\|_{L^2}^2 - \sum_{n=1}^{N} \frac{(f, Y_n)^2}{\|Y_n\|_{L^2}^2} \geq 0$, we have that

(when choosing the "best" c_n)

$$\sum_{n=1}^{N} \frac{(f, Y_n)^2}{\|Y_n\|_{L^2}^2} \leq \|f\|_{L^2}^2 \quad \text{for } \underline{\underline{\text{any } N}}.$$

So we can take the limit $N \to +\infty$ and it still holds!

(assuming $f \in L^2$)

Bessel's inequality: If $\{\varphi_n\}_{n\geq 1}$ is an orthogonal set of functions on (a,b) and $f \in L^2$ on (a,b), then

1) $\displaystyle\sum_{n=1}^{\infty} A_n \varphi_n(x)$ converges in L^2 (not necessarily to f).

$$\left(A_n = \frac{(f, \varphi_n)}{\|\varphi_n\|_{L^2}^2} \right)$$

2) $$\left\| \sum_{n=1}^{\infty} A_n \varphi_n \right\|_{L^2}^2 = \sum_{n=1}^{\infty} A_n^2 \|\varphi_n\|_{L^2}^2 \leq \|f\|_{L^2}^2 .$$

($\{\varphi_n\}$ are orthogonal)

$$\equiv \sum_{n=1}^{\infty} A_n^2 \int_a^b |\varphi_n(x)|^2 \, dx \leq \int_a^b |f(x)|^2 \, dx .$$

Remark: From 1) we see that there may be functions in L^2 that cannot be perfectly approximated by the set $\{\varphi_n\}_{n\geq 1}$ (we could say then that $\{\varphi_n\}_{n\geq 1}$ is not a "basis" of L^2, or that it is not "complete").

<u>Definition</u>: The infinite orthogonal set $\{\varphi_n(x)\}_{n\geq 1}$ is called "complete" in L^2 if

any $f \in L^2$ is perfectly approximated by its "Fourier series" using $\{\varphi_n(x)\}_{n\geq 1}$, i.e.,

if $\forall f \in L^2$, $\quad f = \sum_{n=1}^{\infty} A_n \varphi_n$ in L^2.

$$\left(\text{with} \quad A_n = \frac{(f, \varphi_n)}{\|\varphi_n\|_{L^2}^2}\right)$$

- <u>Theorem-corollary</u>: The orthogonal set $\{\varphi_n\}_{n\geq 1}$ is complete in L^2 if

$$\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |\varphi_n(x)|^2 dx = \int_a^b |f(x)|^2 dx \quad \text{for any } f \in L^2.$$

↳ Parseval's equality.

<u>Proof</u>: Exercise (see page -161-, EN).

→ All these results are stated for <u>any</u> orthogonal set of functions.

→ Going back to our PDE problem, in page -158- -157- we stated that

"The family of eigenfunctions $\{X_n\}$ solving [SL] is complete in L^2

Therefore, for $\{X_n\}$, we can say:

<u>Theorem-corollary</u>: Let $\{X_n\}$ eigenfunctions from [SL]. Then, if $f \in L^2$, we have that

1) $f = \sum_{n=1}^{\infty} A_n X_n$ in L^2

2) Parseval's equality holds: $\int_a^b |f(x)|^2 dx = \sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2 dx$.

V] Pointwise, uniform convergence / Gibbs phenomenon.

We now want to understand the theorems about pointwise and uniform convergence.
While the details are not too important, the ideas in the following two proofs are very common and useful in more advanced maths.

<u>Pointwise convergence</u>: Let $f(x)$ be a C^1 function on $\mathbb{R}$, periodic with period 2π. Then the classical Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$.

<u>Remark</u>: This is the theorem we saw for pointwise convergence but with stronger conditions (to make the proof more simple and neat).

($\to$ The proof for piecewise continuous (f, f') is indeed very similar.)

<u>Proof</u>: [Not easy].

<u>Goal</u>: To prove that for all $x \in \mathbb{R}$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx) \right) \text{ with}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \cos(ny)\, dy \,, \quad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y) \sin(ny)\, dy$$

That is, to prove that

$\lim_{N\to\infty} |S_N(x) - f(x)| = 0$, where the partial sums $S_N(x)$ are

$$S_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} \left(A_n \cos(nx) + B_n \sin(nx) \right).$$

→ First idea: Rewrite the partial sums as a convolution.

To do so, plug in the formulas for the coefficients.

$$\cancel{S_N(x) = \int_{-\pi}^{\pi} (1 + \cos}$$

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \sum_{n=1}^{N} \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)\,dy \right)\cos(nx) +$$

$$+ \sum_{n=1}^{N} \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin(ny)\,dy \right)\sin(nx) =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[1 + 2\sum_{n=1}^{N} \left(\cos(ny)\cos(nx) + \sin(ny)\sin(nx) \right) \right] f(y)\,dy =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 + 2\sum_{n=1}^{N} \cos(n(x-y)) \right) f(y)\,dy$$

If we denote

$$\left\| K_N(x) = 1 + 2\sum_{n=1}^{N} \cos(nx) \right\|$$ DIRICHLET KERNEL

then

$$S_N(x) = \frac{1}{2\pi}(K_N * f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y) f(y)\,dy .$$

→ Second idea: This kernel can be explicitly summed.
(we prove this at the end).

$$\left\| K_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin(x/2)} \right\|$$ DIRICHLET KERNEL

Check: $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = 1$.

$\lim_{N\to\infty} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_N(x)|\,dx \right) = +\infty$ ("not easy" → believe it or google (or try it)).

K_N is even.

So we have $(K_N * f = f * K_N)$ (simply change variables and use periodicity).

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y) f(y)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y) f(x-y)\,dy$$

→ <u>Third idea:</u>

We want to show that

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)\big(f(x-y) - f(x)\big)\,dy \to 0 \text{ as } N\to\infty.$$

The idea is to use the highly oscillatory numerator of $K_N(x)$ to obtain cancellations:

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right) \underbrace{\frac{f(x-y)-f(x)}{\sin(y/2)}}_{\text{independent of } N} dy .$$

<u>Q</u>: Is $\displaystyle\lim_{N\to\infty}\int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right) g(y)\,dy = 0$ for "any" $g(x)$?

↳ That is, is $\displaystyle\lim_{N\to\infty}\left(\sin\left(\left(N+\tfrac{1}{2}\right)x\right), g(x)\right) = 0$?

! Well, if we think of $\{\sin((n+\frac{1}{2})x)\}_{n\geq 1}$ as a "basis" of L^2 (maybe it isn't...), then

(but they are)

$\left(\sin\left(\left(N+\tfrac{1}{2}\right)x\right), g(x)\right)$ would be the N-th component of g in that basis.

And we have seen that functions in L^2 are well approximated by orthogonal basis!

⌈ Why is $\{\sin((n+\frac{1}{2})x)\}_{n\geq 1}$ orthogonal?

↳ They are the eigenfunctions of $\left.\begin{array}{l} X''+\lambda X=0 \\ X(0)=0=X'(\pi) \end{array}\right\}$ ⌋

More specifically, Bessel's inequality guarantees that (↳ we could even use Parseval's equality)

$$\sum_{N=1}^{\infty} \frac{\langle \sin((N+\frac{1}{2})x), g(x)\rangle^2}{\|\sin((N+\frac{1}{2})x)\|_{L^2}} \leq \|g\|_{L^2}^2 \quad (\text{as long as } \|g\|_{L^2}<\infty).$$

Since this implies that it is a convergent series, it necessarily holds that the terms tend to zero:

$$\left(\sin((N+\tfrac{1}{2})x), g(x)\right) \to 0 \quad \text{as } N\to 0 \quad \|.$$

• We need to check that our "$g(x)$" is indeed in L^2:

$$g(y)=\frac{f(x-y)-f(x)}{\sin(y/2)} \to \text{Continuous if } f\in C^1, \text{ thus } g\in L^2 /\!/.$$

⌈ $\|g\|_{L^2}^2 = \int_{-\pi}^{\pi} \frac{|f(x-y)-f(x)|^2}{(y/2)^2}\frac{(y/2)^2}{\sin^2(y/2)}\,dy \leq 4\max_{[-\pi,\pi]}|f'(x)|\int_{-\pi}^{\pi}\frac{(y/2)^2}{\sin^2(y/2)}\,dy < +\infty$ ⌋