MATH 425

LECTURE 19

Last day:

If $\{Y_n\}_{n\geq 1}$ orthogonal set of functions, $f \in L^2$ and

$A_n = \dfrac{(f, Y_n)}{\|Y_n\|_{L^2}^2}$ (Fourier coefficients), then

• Bessel's inequality: $\sum_{n=1}^{\infty} A_n^2 \|Y_n\|_{L^2}^2 \leq \|f\|_{L^2}^2$.

If in addition the orthogonal set is complete, like the eigenfunctions $\{X_n\}_{n\geq 1}$ coming from $X'' + \lambda X = 0$ with symmetric BC, then

• Parseval's equality: $\sum_{n=1}^{\infty} A_n^2 \|X_n\|_{L^2}^2 = \|f\|_{L^2}^2$.

We ended the lecture proving the pointwise convergence of Fourier series for a 2π-periodic C^1 function $\rfloor$

(Th: Pointwise convergence)

$f \in C^1$ on $\mathbb{R}$, 2π periodic $\Rightarrow$ $\underbrace{\frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx)\right)}_{\lim_{N\to\infty} S_N(x)} = f(x) \quad \forall x \in \mathbb{R}$.

Proof.

Goal: $\lim_{N\to\infty} \left| S_N(x) - f(x) \right| = 0$ (for all $x \in \mathbb{R}$).

Steps:

1) Introduce the Dirichlet Kernel: Substitute the coefficients A_n, B_n by its formulas,

$$S_N(x) = \frac{1}{2\pi} (k_N * f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k_N(x-y) f(y)\, dy =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} k_N(y) f(x-y)\, dy, \text{ with}$$

$$k_N(x) = 1 + 2 \sum_{n=1}^{N} \cos(nx).$$

2) Recall that this Kernel can be written as (proof at the end).

$$k_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin(x/2)} \qquad \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} k_N(x)\, dx = 1 \right]$$

3) Show that

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y)\big(f(x-y) - f(x)\big)\,dy \to 0 \quad \text{as } N\to\infty.$$

Notice that

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(\left(N+\tfrac{1}{2}\right)y\right)\frac{f(x-y)-f(x)}{\sin(y/2)}\,dy.$$

The second term is independent of N. If we think of x as a parameter, let's denote

$$g_x(y) = \frac{f(x-y)-f(x)}{\sin(y/2)}.$$

With this notation, $S_N(x) - f(x) = \frac{1}{2\pi}\left(\sin\left(\left(N+\tfrac{1}{2}\right)y\right), g_x(y)\right)$.

Thus the question now is:

$$\left(\sin\left(\left(N+\tfrac{1}{2}\right)y\right), g_x(y)\right) \to 0 \quad \text{as } N\to\infty \; ?$$

Recall that $\{\sin((n+\frac{1}{2})x)\}_{n\geq 1}$ is an orthogonal set of functions on $(-\pi,\pi)$, so we can use Bessel's inequality:

$$\sum_{N=1}^{\infty} \frac{\left(\sin((N+\frac{1}{2})y), g_x(y) \right)^2}{\| \sin((N+\frac{1}{2})y) \|_{L^2}^2} \leq \| g_x \|_{L^2}^2 \quad (\text{if } g_x \in L^2).$$

From here, since $\int_{-\pi}^{\pi} |\sin((N+\frac{1}{2})y)|^2 \, dy = \pi$, we can

conclude that $(\sin((N+\frac{1}{2})y), g_x(y)) \to 0$ as $N \to \infty$ //.

- The only thing left to check is that $g_x \in L^2$ (for any $x \in \mathbb{R}$).

But, indeed

$$g_x(y) = \frac{f(x-y) - f(x)}{\sin(y/2)}, \text{ thus}$$

$$\| g_x \|_{L^2}^2 = \int_{-\pi}^{\pi} \frac{|f(x-y) - f(x)|^2}{(y/2)^2} \frac{(y/2)^2}{\sin^2(y/2)} \, dy \leq 4 \max_{[-\pi,\pi]} |f'(x)| \left(\int_{-\pi}^{\pi} \frac{(y/2)^2}{\sin^2(y/2)} \, dy \right) < \infty //.$$

- We used in the proof that

$$K_N(x) = 1 + 2 \sum_{n=1}^{N} \cos(nx) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}.$$

Let's show it:

Recall that $\cos(nx) = \dfrac{e^{inx} + e^{-inx}}{2}$. Then,

$$1 + 2\sum_{n=1}^{N} \cos(nx) = 1 + \sum_{n=1}^{N} \left(e^{inx} + e^{-inx}\right) = \sum_{n=-N}^{N} e^{inx}.$$

That is,

$$K_N(x) = e^{-iNx} + e^{-i(N-1)x} + \dots + 1 + \dots + e^{iNx}, \text{ and thus}$$

$$e^{ix} K_N(x) = e^{-i(N-1)x} + \dots + 1 + \dots + e^{iNx} + e^{i(N+1)x}$$

$\Rightarrow$

$$\Rightarrow K_N(x) - e^{ix} K_N(x) = e^{-iNx} - e^{i(N+1)x} \Rightarrow$$

$$\Rightarrow K_N(x) = \frac{e^{-iNx} - e^{i(N+1)x}}{1 - e^{ix}} \underset{\cdot e^{-ix/2}}{=} \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-ix/2} - e^{ix/2}} = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin(x/2)} \,//.$$

- Let's finish the chapter of Fourier series with a proof for uniform convergence.

<u>Theorem</u>: (Unif. convergence of classical (full) Fourier series)

$f(x)$ continuous, 2π-periodic
$f'(x)$ piecewise continuous $\Rightarrow$ ((Classical full) Fourier series converges uniformly.

// <u>Proof</u>: Let $S_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} \left(A_n \cos(nx) + B_n \sin(nx) \right)$.

<u>Goal</u>: $\lim_{N\to\infty} \max |f(x) - S_N(x)| = 0$.

- First of all, we know that the conditions guarantee that the Fourier series converges pointwise to $f(x)$.

- We are going to show first that the coefficients decay fast enough (this allows us to conclude the proof with basically the argument of Weierstrass's test).

- Denote A_n, B_n the Fourier coefficients of $f(x)$ and A'_n, B'_n the ones of $f'(x)$. Then, using integration by parts:

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) = \frac{1}{n\pi} f(x) \cancel{\sin(nx)}\Big]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx = -\frac{1}{n} B'_n.$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \dots = \frac{1}{n} A'_n$$

Since $f'(x)$ is piecewise-continuous on $[-\pi,\pi]$, it is in $L^2_{([-\pi,\pi])}$,
thus Bessel's inequality yields that

$$\sum_{n=1}^{\infty}\left(|A'_n|^2+|B'_n|^2\right)<\infty.$$

Therefore,

here we need the "smoothness" of f.

$$\sum_{n=1}^{\infty}\left(|A_n\cos(nx)|+|B_n\sin(nx)|\right)\le\sum_{n=1}^{\infty}\left(|A_n|+|B_n|\right)=$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}\left(|B'_n|+|A'_n|\right)\le\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left(|B'_n|+|A'_n|\right)^2\right)^{1/2}\le$$

Schwarz's ineq.

$$\le\left(\frac{\pi^2}{6}\right)^{1/2}\left(\sum_{n=1}^{\infty}2\left(|A'_n|^2+|B'_n|^2\right)\right)^{1/2}<\infty.$$

$(a+b)^2\le(a^2+b^2)\cdot 2$

- By the Weierstrass' comparison test, the Fourier converges uniformly (to $f(x)$, since we know this is the pointwise limit).

(In the book you can see how the proof concludes)

<u>Remark</u>: It is important to realize that the smoothness of a function is directly connected to the decay of its Fourier coefficts.

$\boxed{\text{Chapter 6}}$: Laplace and Poisson equations.

$$\| \Delta u = f \|$$

- $f \equiv 0 \rightarrow \Delta u = 0$ Laplace equation. (u is said to be harmonic)
- $f \neq 0 \rightarrow \Delta u = f$ Poisson equation.

$$\left(\Delta u = \sum_{i=1}^{n} \partial_{x_i}^2 u \right)$$

- Motivation
  - Physics: Potential Theory (Electrostatics, fluids...)
    - ↳ Green's function (Chapt. 7).
  - Mathematics: → (PDEs)
    - → Complex variables.

- Goal: Solve $\begin{cases} \Delta u = f & \text{in a domain } D \in \mathbb{R}^n \\ BC \text{ on } \partial D & (\text{Dirichlet, Neumann, Robin}). \end{cases}$

- Some important properties: → maximum principle.
  → Invariance to rotations.

I] Maximum principle.

Let $D \subseteq \mathbb{R}^2$ be an open connected bounded set.

Let $u(x,y)$ solve Laplace equation (i.e. u is harmonic), and continuous on $\bar{D} = D \cup \partial D$.

Then,
$$\max_{\vec{x} \in \bar{D}} u(\vec{x}) = \max_{\vec{x} \in \partial D} u(\vec{x}) . \quad \text{(weak)}$$

Moreover, (unless u is constant) the maximum is only attained on ∂D.
(strong)

[analogously for the minimum].

Proof: (Similar to the heat equation).

↱ We want to obtain a contradiction: if $u(x,y)$ attains a maximum inside D, then $\Delta u = u_{xx} + u_{yy} \le 0$. If the inequality was strict, we would have finished !!.

- Define $v(\vec{x}) = u(\vec{x}) + \varepsilon |\vec{x}|^2$. Then,

$$\Delta v = \Delta u + \varepsilon \Delta (x_1^2 + x_2^2) = \Delta u + 4\varepsilon = 4\varepsilon > 0 \text{ in } D.$$

- However, if $\vec{x}^*$ is a maximum point of $v(\vec{x})$ in the interior of D, then $\Delta v \le 0$ ↯ $\Rightarrow$ So $v(\vec{x})$ has no interior maximum in D.

• Since $\vec{v}(\vec{x})$ is continuous and $\bar{D}$ is closed and bounded, then $v(\vec{x})$ has to attain its maximum somewhere in $\bar{D}$, thus in ∂D.

Denote $\vec{x}^* \in \partial D$ a point where v is maximum.

Then,

$\cancel{u(\vec{x}^*) \le v(\vec{x}^*) =}$

$$u(\vec{x}) \le v(\vec{x}) \le v(\vec{x}^*) = u(\vec{x}^*) + \varepsilon |\vec{x}^*|^2 \le$$

$$\le \max_{\vec{x} \in \partial D} u(\vec{x}) + \varepsilon |\vec{x}^*|^2 \le \max_{\partial D} u + \varepsilon L^2 ,$$

with L the maximum distance from ∂D to the origin, i.e.,

$$L = \max_{\vec{x} \in \partial D} |\vec{x}| .$$

• Finally, letting $\varepsilon \to 0$, $\quad u(\vec{x}) \le \max_{\partial D} u \quad \forall \vec{x} \in D$.

That is,

$$u(\vec{x}) \le u(\tilde{\vec{x}}) \quad \forall \vec{x} \in \bar{D} \text{ and some } \tilde{\vec{x}} \in \partial D \} //.$$

__Remark__: We have only proved the weak maximum principle.

__Remark__: As usual, it is straightforward to prove uniqueness of solutions using the maximum (and minimum) principle }

II) Rotational invariance.

If we consider a rotation of the variables, the Laplacian does not change:

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \to \begin{aligned}\tilde{x} &= x\cos\alpha + y\sin\alpha\\ \tilde{y} &= -x\sin\alpha + y\cos\alpha\end{aligned}$$

Chain rule:

$$u_x = u_{\tilde{x}}\cos\alpha - u_{\tilde{y}}\sin\alpha$$

$$u_y = u_{\tilde{x}}\sin\alpha + u_{\tilde{y}}\cos\alpha$$

$$u_{xx} = (u_{\tilde{x}}\cos\alpha - u_{\tilde{y}}\sin\alpha)_{\tilde{x}}\cos\alpha - (u_{\tilde{x}}\cos\alpha - u_{\tilde{y}}\sin\alpha)_{\tilde{y}}\sin\alpha =$$

$$= u_{\tilde{x}\tilde{x}}\cos^2\alpha - u_{\tilde{x}\tilde{y}}\sin\alpha\cos\alpha - u_{\tilde{x}\tilde{y}}\cos\alpha\sin\alpha + u_{\tilde{y}\tilde{y}}\sin^2\alpha,$$

$$u_{yy} = (u_{\tilde{x}}\sin\alpha + u_{\tilde{y}}\cos\alpha)_{\tilde{x}}\sin\alpha + (\quad)_{\tilde{y}}\cos\alpha =$$

$$= u_{\tilde{x}\tilde{x}}\sin^2\alpha + u_{\tilde{y}\tilde{x}}\cos\alpha\sin\alpha + u_{\tilde{x}\tilde{y}}\sin\alpha\cos\alpha + u_{\tilde{y}\tilde{y}}\cos^2\alpha \to$$

$$\Rightarrow \boxed{\Delta u = u_{xx} + u_{yy} = u_{\tilde{x}\tilde{x}} + u_{\tilde{y}\tilde{y}} = \tilde{\Delta} u}.$$

Exercise: (Review) Check that in polar coordinates,

$$\Delta u = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} .$$

Also, find solutions to Laplace equation which are themselves rotational invariant (i.e., $\partial_\theta u = 0$).

Exercise: Find rotational invariant solutions to $\Delta u = 0$ in thee dimensions (use the formula for Δ in spherical coordinates from the book).

III) Laplace equation on rectangles.

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 \quad \text{in } D = \{0 < x < a, 0 < y < b\} \\ BC \text{ on } \quad x = 0,\ x = a, \\ \qquad\qquad y = 0,\ y = b \end{cases}$$

Method: Separation of variables.

① If there are inhomogeneous BC in more than one side, split the problem using linearity.

For example:

$$\left.\begin{aligned} &\Delta u = 0 \quad 0 < x < a,\ 0 < x < b \\ &u(x,y) = 0, \quad x = 0 \\ &u(x,y) = f(x), \quad y = 0 \\ &u_x(x,y) = g(y), \quad x = a \\ &u(x,y) = 0, \quad y = b \end{aligned}\right\} \to \left\{\begin{aligned} &\text{Write } u = u_1 + u_2, \text{ where} \\ &\Delta u_1 = 0 = \Delta u_2 \\ &u_1(x,y) = 0 = u_2(x,y), \quad x = 0 \\ &u_1(x,y) = f(x),\ u_2(x,y) = 0, \quad y = 0 \\ &u_{1_x}(x,y) = 0,\ u_{2_x}(x,y) = g(y), \quad x = a \\ &u_1(x,y) = 0 = u_2(x,y) = 0, \quad y = b \end{aligned}\right.$$

So we'd need to solve two problems.

② (On each of the problems) separate variables.

③ Use the homogeneous BC to get the eigenvalues.

④ After summing the series, plug in the inhomogeneous conditions, which will require finding a Fourier series expansion.

<u>Example:</u>