MATH 425

LECTURE 23

We finish last day with the statement of a theorem,

- **Theorem:** Heisenberg's inequality.

  Let $f \in L^2$, $x f(x) \in L^2$, $f' \in L^2$. Then,

  $$\| x f(x) \|_{L^2}^2 \, \| f' \|_{L^2}^2 \geq \frac{1}{4} \| f \|_{L^2}^4$$

To prove it, let's recall first some things.

We need to work with inner products using complex numbers,

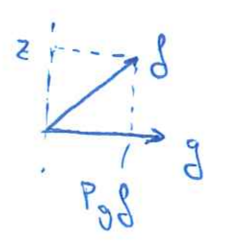
$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx \in \mathbb{C} \qquad (\text{but } x \in \mathbb{R}).$$

Anyway, Cauchy-Schwarz inequality is still true:

$$| \langle f, g \rangle | \leq \| f \|_{L^2} \| g \|_{L^2}.$$

Proof: Let $z = f - P_g f = f - \frac{\langle f, g \rangle}{\langle g, g \rangle} g$. Then,

$$\langle z, g \rangle = \langle f, g \rangle - \frac{\langle f, g \rangle}{\langle g, g \rangle} \langle g, g \rangle = 0 \Rightarrow z \text{ is orthogonal to } g.$$



Since $f = z + \frac{\langle f, g\rangle}{\langle g, g\rangle} g$, we have that

$$\langle f, f\rangle = \langle z + \frac{\langle f, g\rangle}{\langle g, g\rangle} g, z + \frac{\langle f, g\rangle}{\langle g, g\rangle} g\rangle =$$

$$= \langle z, z\rangle + \left(\frac{\langle f, g\rangle}{\langle g, g\rangle}\right)^2 \langle g, g\rangle .$$

[This is nothing more than Pythagoras th.]

where we have used $\langle z, g\rangle = 0$.
That is, we have that

$$\|f\|_{L^2}^2 = \underbrace{\|z\|^2}_{\geq 0} + \frac{\langle f, g\rangle^2}{\|g\|_{L^2}^4} \|g\|_{L^2}^2 \Rightarrow$$

$$\Rightarrow \|f\|_{L^2}^2 \geq \frac{\langle f, g\rangle^2}{\|g\|_{L^2}^2} \Rightarrow \|f\|_{L^2}^2 \|g\|_{L^2}^2 \geq \langle f, g\rangle^2 \Rightarrow$$

$$\Rightarrow \|f\|_{L^2} \|g\|_{L^2} \geq |\langle f, g\rangle| .$$

<u>Remark</u>: The proof is valid for any vector space with inner product (and norm defined by $\|\cdot\| = \langle \cdot, \cdot\rangle^{1/2}$).

- We will also use that for any $y, z \in \mathbb{C}$,

$$\begin{cases} z + \bar{z} = 2\,\mathrm{Re}(z) \\ |\mathrm{Re}(z)| \leq |z| . \end{cases} , \quad \overline{yz} = \bar{y}\bar{z}, \quad \bar{\bar{z}} = z .$$

Proof of H. ineq.

- Apply Cauchy-Schwartz inequality to $x f(x)$ and $f'(x)$:

$$\|x f(x)\|_{L^2}^2 \|f'\|_{L^2}^2 \geq |\langle x f(x), f'(x)\rangle|^2 = \left|\int_{-\infty}^{\infty} x f(x) \overline{f'(x)}\, dx\right|^2 \geq$$

$$\geq \left| \mathrm{Re} \int_{-\infty}^{\infty} x f(x) \overline{f'(x)}\, dx \right|^2 = \left| \int_{-\infty}^{\infty} \mathrm{Re}\left(x f(x) \overline{f'(x)}\right) dx \right|^2 =$$

$$= \left| \frac{1}{2} \int_{-\infty}^{\infty} \left(x f(x) \overline{f'(x)} + x \overline{f(x)} f'(x) \right) dx \right|^2 = \frac{1}{4} |I|^2 .$$

- Let's compute the last integral integrating by parts:

$$I = -\int_{-\infty}^{\infty} \frac{d}{dx}(x f(x)) \overline{f(x)}\, dx + \underbrace{x \cancel{f(x)} \overline{f(x)}\Big]_{-\infty}^{\infty}}_{=0} + \quad \text{because } x f(x) \in L^2$$

$$-\int_{-\infty}^{\infty} \frac{d}{dx}\left(x \overline{f(x)}\right) f(x)\, dx + \underbrace{x \overline{f(x)} \cancel{f(x)}\Big]_{-\infty}^{\infty}}_{=0} =$$

$$= -\int_{-\infty}^{\infty} \left(f(x) \overline{f(x)} + x f'(x) \overline{f(x)} + \overline{f(x)} f(x) + x \overline{f'(x)} f(x) \right) dx =$$

$$= -2 \|f\|_{L^2}^2 - I \Rightarrow$$

$$\Rightarrow I = -\|f\|_{L^2}^2 .$$

• Going back to the first part the proof is concluded:

$$\|x f(x)\|_{L^2}^2 \|f'\|_{L^2}^2 \geq \frac{1}{4} |I|^2 = \frac{1}{4} \|f\|_{L^2}^4 \,//.$$

<u>Remark</u>: The inequality is very often written as follows

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 \frac{d\omega}{2\pi} \right) \geq \frac{1}{4} \|f\|_{L^2}^4 .$$

[Simply recall Plancherel's equality:

$$\|f'\|_{L^2}^2 = \frac{1}{2\pi} \|\widehat{f'}\|_{L^2}^2 = \frac{1}{2\pi} \|i\omega \hat{f}(\omega)\|_{L^2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega \Big] .$$

• <u>Remark</u>: Connection to Heisenberg's uncertainty principle.

According to quantum mechanics, the position and the momentum of a particle (that moves in one dimension) can be described by its "state function" $\varphi(x,t)$. This is governed by Schrödinger equation, and it is complex-valued.

In particular, this state function defines probability distributions for position and momentum:

Position: $P_\varphi(a < x < b) = \int_a^b |\varphi(x,t)|^2 dx$

Momentum: $P_\varphi(c < \omega < d) = \int_c^d |\hat{\varphi}(\omega, t)|^2 \frac{d\omega}{2\pi}$

[Why the momentum is given by the Fourier transform of the state function is for us an statement from physics too].

- Of course, for those to be well-defined probabilities, it is required that

$\int_{-\infty}^{\infty} |\varphi(x)|^2 dx = 1$ (i.e. $|\varphi|^2$ is a probability density function).

$\left[\Rightarrow \int_{-\infty}^{\infty} |\hat{\varphi}(\omega,t)|^2 \frac{d\omega}{2\pi} = 1 \right]$.

- Now, recall the definition for expected value and variance:

$$E_\varphi[x] = \int_{-\infty}^{\infty} x\,|\varphi(x)|^2\,dx \,, \quad E_\varphi[\omega] = \int_{-\infty}^{\infty} \omega\,|\hat{\varphi}(\omega)|^2\,\frac{d\omega}{2\pi} \,.$$

$$V_\varphi[x] = E_\varphi\left[(x - E_\varphi[x])^2\right] = \int_{-\infty}^{\infty} (x - E_\varphi[x])^2\,|\varphi(x)|^2\,dx \,,$$

$$V_\varphi[\omega] = \dots \qquad = \int_{-\infty}^{\infty} (\omega - E_\varphi[\omega])^2\,|\hat{\varphi}(\omega)|^2\,\frac{d\omega}{2\pi} \,.$$

Notice that if $E_\varphi[x] = 0 = E_\varphi[\omega]$, then our previous Heisenberg's inequality would give that

$$\underbrace{\left\| V_\varphi[x]\,V_\varphi[\omega] \geq \frac{1}{4} \right\|} \qquad \left(\left(\int_{-\infty}^{\infty} x^2\,|\varphi(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \omega^2\,|\hat{\varphi}(\omega)|^2\,\frac{d\omega}{2\pi}\right) \geq \frac{1}{4}\underbrace{\|\varphi\|_{L^2}^4}_{=1} \right)$$

Heisenberg's uncertainty principle: The variance of position and momentum cannot be arbitrarily small at the same time.

Remark: We have added an artificial condition, namely, that $E_\varphi[x] = 0$, $E_\varphi[\omega] = 0$.

To "avoid" proving Heisenberg's inequality in the case these are not zero, we will prove the following lemma:

(we are indeed proving it through the lemma).

• **Lemma**: Let $\varphi(x)$ be a state function with

$E_\varphi[x] = a$, $E_\varphi[\omega] = b$. Then,

1) $\psi(x) = e^{-ibx}\varphi(x+a)$ is a state function with

2) $E_\psi[x] = 0 = E_\psi[\omega]$ and with the same variances,

3) $V_\psi[x] = V_\varphi[x]$, $V_\psi[\omega] = V_\varphi[\omega]$.

Remark: The condition "state function" could more generally be replaced by $\varphi \in L^2$ (and then one would obtain that $\psi \in L^2$ with $\|\psi\|_{L^2} = \|\varphi\|_{L^2}$).

Proof:

1) $$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} \underbrace{|e^{-2ibx}|}_{=1} |\varphi(x+a)|^2 dx \underset{y = x+a}{=} \int_{-\infty}^{\infty} |\varphi(y)|^2 dy = 1.$$

2) • $$E_\psi[x] = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx = \int_{-\infty}^{\infty} x\,|e^{-ibx}\varphi(x+a)|^2 dx = \int_{-\infty}^{\infty} x\,|\varphi(x+a)|^2 dx \underset{y=x+a}{=}$$

$$= \int_{-\infty}^{\infty} (y-a)|\varphi(y)|^2 dy = \int_{-\infty}^{\infty} y|\varphi(y)|^2 dy - a = 0.$$

- $E_\psi[\omega] = \int_{-\infty}^{\infty} \omega \, |\hat{\psi}(\omega)|^2 \, d\omega \, \frac{1}{2\pi}$,

$$\left[\begin{aligned} \hat{\psi}(\omega) &= \mathcal{F}\left(e^{-ibx}\varphi(x+a)\right)(\omega) = \mathcal{F}\left(\varphi(x+a)\right)(\omega+b) = \\ &= e^{-ia(\omega+b)}\,\hat{\varphi}(\omega+b) \qquad \text{(two traslations)} \end{aligned}\right]$$

$$E_\psi[\omega] = \int_{-\infty}^{\infty} \omega \, \underbrace{|e^{-ia(\omega+b)}|^2}_{=1} \, |\hat{\varphi}(\omega+b)|^2 \frac{d\omega}{2\pi} \underset{y=\omega+b}{=} \int_{-\infty}^{\infty} (y-b)\,|\hat{\varphi}(y)|^2 \frac{dy}{2\pi} =$$

$$= \int_{-\infty}^{\infty} y\,|\hat{\varphi}(y)|^2 \frac{dy}{2\pi} - b = 0$$

Finally, we compute the variances:

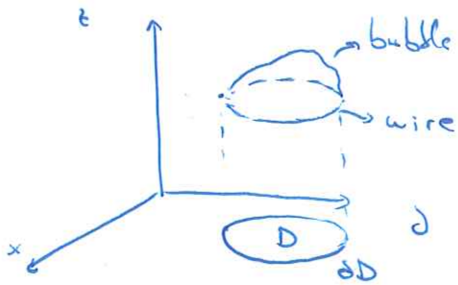
$$V_\psi[x] = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx = \int_{-\infty}^{\infty} x^2 |\varphi(x+a)|^2 dx \underset{y=x+a}{=} V_\varphi[x]$$

$$V_\psi[\omega] = \int_{-\infty}^{\infty} \omega^2 |\hat{\psi}(\omega)|^2 dx = \int_{-\infty}^{\infty} \omega^2 |\hat{\varphi}(\omega+b)|^2 \frac{d\omega}{2\pi} = V_\varphi[\omega].$$

• Calculus of variations (Section 14.3)

Motivation: Shape of a soap bubble

We want to find the shape of a soap bubble spanned by a wire ring:



According to physics, the shape is that one that minimices the "surface energy", which is proportional to the area of the surface.

Thus, our goal is to find the surface with minimum are satisfying the boundary condition given by the wire ring.

Assuming we only consider parametrized surfaces in the form of a graph $z = u(x,y)$, the problem is written as:

$$\left.\begin{aligned} \min \quad & A = \iint_D \sqrt{1+u_x^2+u_y^2}\,dx\,dy \\ \text{s.t.} \quad & u(x,y) = h(x,y) \text{ on } \partial D \end{aligned}\right\}$$

($z = h(x,y)$ is the curve describing the wire ring).

$(x, y, z(x,y)) \rightsquigarrow$ tangents: $\vec{t}_x = (1,0,z_x)$, $\vec{t}_y = (0,1,z_y)$.

↳ Normal:
$\vec{t}_x \times \vec{t}_y = (-z_x, -z_y, 1) \longrightarrow \|\vec{t}_x \times \vec{t}_y\| = \sqrt{1+z_x^2+z_y^2}$

- So we have an optimization problem, but the set where the "variables" lives is "infinite-dimensional" (functions).

Review from calculus: If $f: U \subset \mathbb{R}^n \to \mathbb{R}$, their max/min happen at critical points:

$\| x \,/\, Df(x) = 0 \|$ Here $Df(x)$ is the derivative matrix of f at x (for example, the gradient if $n=2$).

$\downarrow$ equivalent to

$Df(x) \cdot z = 0 \quad \forall z \in \mathbb{R}^n$

$\downarrow$ equivalent to (simply apply the chain rule below)

$$\left\| \frac{d}{d\varepsilon} f(x+\varepsilon z)\Big|_{\varepsilon=0} = 0 \quad \text{for all } z \in \mathbb{R}^n \right\| \quad (*)$$

This last characterization of critical points is easily adapted to allow z a function.

- In the "calculus of variations", the goal is to find the critical points of a functional instead of a function. For example,

$$\left.\begin{aligned} E : \text{"functions"} &\longrightarrow \mathbb{R} \\ u &\longmapsto E[u] = \int_a^b F(x, u, u')\,dx \end{aligned}\right\} \quad (1)$$

<u>Remark</u>: The domain of E has to be specified. In particular, one includes there the boundary conditions.

- Let's find the critical points of $E[u]$ in (1) when the domain are (smooth) functions whose values at $x = a$, $x = b$ are given.

We generalize formula (*):

$$\frac{d}{d\varepsilon} E[u + \varepsilon v]\Big|_{\varepsilon = 0} = 0$$

for all $v(x)$ (smooth) function that vanishes at $x = a$, $x = b$. (so that $u + \varepsilon v$ is in the domain of E).

$$\frac{d}{d\varepsilon}\left(\int_a^b F(x, u + \varepsilon v, u' + \varepsilon v')\,dx\right)\Big|_{\varepsilon=0} =$$

(Denote $F = F(x, u, p)$)

(only to clarify notation)

chain rule.

$$= \int_a^b \left(\frac{\partial F}{\partial u} v + \frac{\partial F}{\partial p} v'\right) dx = 0 .$$

Finally, integrate by parts: (use that $v(a) = v(b) = 0$)

$$\int_a^b \left(\frac{\partial F}{\partial u} v - \frac{d}{dx}\left(\frac{\partial F}{\partial p}\right) v\right) dx = 0 \quad \forall\, v(x) \longrightarrow$$

$$\to \left\| \frac{\partial F}{\partial u} = \frac{d}{dx} \frac{\partial F}{\partial p} \right.$$

- This is the Euler-Lagrange equation, an ODE that any "critical point" $u(x)$, (with values fixed at the ends) of the functional $E[u]$ in (1) must satisfy

$$\left\| \frac{\partial F}{\partial u}(x, u(x), u'(x)) = \frac{d}{dx}\left(\frac{\partial F}{\partial p}(x, u(x), u'(x)) \right) \right\| \quad \text{Euler-Lagrange}$$

Notice that F is a function of three variables given as data, so the only unknown is $u(x)$.

- Remark: This is still not sufficient for our bubble problem. There we had $u(x,y)$, functions of two variables.

↳ But the idea is the same:

$$E[u] = \iint_D F(x, y, u, u_x, u_y)\, dx\, dy \quad (\text{domain: } u \text{ fixed at } \partial D).$$

Critical points:

$$\frac{d}{d\varepsilon} E[u + \varepsilon v] \Big|_{\varepsilon = 0} = 0 \quad \text{for all } v \text{ such that } v \equiv 0 \text{ on } \partial D.$$

Denote $F = F(x, y, u, p, q)$, so that we obtain

$$\iint_D \left(\frac{\partial F}{\partial u} v + \frac{\partial F}{\partial p} v_x + \frac{\partial F}{\partial q} v_y \right) dx\,dy = 0 =$$

$$= \iint_D \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \frac{\partial F}{\partial p} - \frac{\partial}{\partial y} \frac{\partial F}{\partial q} \right) v \, dx\,dy = 0 \quad \forall v \Rightarrow$$

$$\Rightarrow \frac{\partial F}{\partial u} = \frac{\partial}{\partial x} \frac{\partial F}{\partial p} + \frac{\partial}{\partial y} \frac{\partial F}{\partial q} \rightsquigarrow$$

$$\rightsquigarrow \left\| \frac{\partial F}{\partial u}(x, y, u, u_x, u_y) = \frac{\partial}{\partial x} \frac{\partial F}{\partial p}(x, y, u, u_x, u_y) + \frac{\partial}{\partial y} \frac{\partial F}{\partial q}(x, y, u, u_x, u_y) \right\|$$

$\hookrightarrow$ PDE with unknow $u(x, y)$.

- We can apply this to the soap bubble in -9-:

$$F = F(u_x, u_y) = \sqrt{1 + u_x^2 + u_y^2}$$

$$\hookrightarrow \frac{\partial F}{\partial u} \equiv 0, \quad \frac{\partial F}{\partial (u_x)} = \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}, \quad \frac{\partial F}{\partial (u_y)} = \frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}$$

So the Euler-Lagrange equation is

$$\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}} \right)_x + \left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}} \right)_y = 0 \quad \Big\|$$ Minimal Surface equation.

Solutions to this PDE gives the shape $z = u(x, y)$ of the bubbles.