Summary of the course

- 1st order linear/quasilinear PDE → Characteristics. / → Coordinate method.
  ↓
  Fourier transform   Chap 1

- Linear PDEs in $\mathbb{R}$ (whole line)   Chap 2, 3

  → Heat equation:
  - max. principle / energy methods } → uniqueness.
  - heat kernel → solution (homogeneous/inhomogeneous).
    ↑ Fourier Transform.

  → Wave equation:
  - energy method → uniqueness.
  - D'Alembert + Green's/characteristics } → solution (homogeneous/inhomogeneous).
    ↑ Fourier transform.

- Linear PDE in $\mathbb{R}^+$ (half-line)   Chap. 3
  (*) ↳ odd/even extensions (for both homogeneous/inhomogeneous).
  - odd: Dirichlet BC
  - even: Neumann BC

(*) For us, only the heat equation (and variations of it).

• Linear PDE in finite intervals : separation of variables

(Heat, wave, Laplace equation).

$\downarrow$

Fourier series (eigenvalues).

Chap. 4, 5, 6.

(squares / circles
boxes)

$\downarrow$

(Poisson's kernel ...)

⇢ (Dirichlet kernel
completeness ...)

+ Calculus of variation.

+ Fourier transform ($\rightarrow$ solves linear PDE without boundaries)

MATH 425
LECTURE 25 : Review

$\boxed{P1}$ Solve $\left.\begin{array}{l} u_t = 2u_x + 3u \\ u(x,0) = f(x) \end{array}\right\}$ using

1) Fourier
2) Characteristics
3) Coordinate method.

Solution:

1) $\left.\begin{array}{l} \hat{u}_t = (2i\omega + 3)\hat{u} \\ \hat{u}(\omega,0) = \hat{f}(\omega) \end{array}\right\} \Rightarrow \hat{u}(\omega) = \hat{f}(\omega) e^{(2i\omega+3)t} = e^{3t}\hat{f}(\omega)e^{2i\omega t}$,

so taking inverse Fourier transform,

$$u(x,t) = e^{3t}\mathcal{F}^{-1}\left(\hat{f}(\omega)e^{2i\omega t}\right)(x) = e^{3t}f(x+2t).$$
$$\uparrow$$
$$\mathcal{F}\left(f(x-a)\right)(\omega) = e^{-ia\omega}\hat{f}(\omega)$$

2) Characteristics: $u_t - 2u_x = 3u$

$\hookrightarrow \left.\begin{array}{l} \frac{dx}{ds}(s) = -2 \rightarrow x(s) = -2s + c_1 \\ \frac{dt}{ds}(s) = 1 \rightarrow t(s) = s + c_2 \end{array}\right\}$ The PDE becomes

$$\frac{d}{ds}\left(u(x(s),t(s))\right) - 3u(x(s),t(s)) = 0,$$

with $u(x(0),t(0)) = u(c_1,c_2) \underset{\substack{\uparrow \\ \text{Take } c_2=0}}{=} u(c_1,0) = f(c_1)$.

That is,

$$\left.\begin{aligned} u'(s) - 3u(s) = 0 \\ u(0) = f(c_1) \end{aligned}\right\} \to u(s) = f(c_1)e^{3s}.$$

Finally, substitute back from s, c_1 to x, t:

$$\left.\begin{aligned} x &= -2s + c_1 \\ t &= s \end{aligned}\right\} \to \begin{aligned} s &= t \\ c_1 &= x + 2t \end{aligned}, \text{ so } u(x,t) = f(x+2t)e^{3t} //.$$

3) New variables:

$$\left.\begin{aligned} \tilde{x} &= t - 2x \\ \tilde{t} &= 2t + x \end{aligned}\right\} \text{ By the chain rule, } \left.\begin{aligned} u_x &= -2u_{\tilde{x}} + u_{\tilde{t}} \\ u_t &= u_{\tilde{x}} + 2u_{\tilde{t}} \end{aligned}\right| \text{, so the PDE becomes}$$

$$u_{\tilde{x}} + \cancel{2u_{\tilde{t}}} = -4u_{\tilde{x}} + \cancel{2u_{\tilde{t}}} + 3u \to 5u_{\tilde{x}} - 3u = 0 \to$$

$$\to u(\tilde{x},\tilde{t}) = e^{-\frac{3}{5}\tilde{x}} c(\tilde{t}) \to$$

$$\to u(x,t) = e^{-\frac{3}{5}(t-2x)} c(2t+x)$$

Initial data: $u(x,0) = f(x) = e^{\frac{6}{5}x} c(x) \to c(x) = e^{-\frac{6}{5}x} f(x)$,

thus,

$$u(x,t) = e^{-\frac{6}{5}(2t+x)} f(2t+x) e^{-\frac{3}{5}(t-2x)} = e^{-\frac{15}{5}t} f(2t+x) = e^{-3t} f(2t+x) //.$$

$\boxed{P2}$ Let $f_a(x) = \dfrac{a}{\pi(x^2+a^2)}$, $a>0$.

1) Show that $f_a(x)$ is a probability density function, that is, $f_a(x) \geq 0$, $\int_{-\infty}^{\infty} f_a(x)\,dx = 1$.

2) The probability density function of the sum of two random variables is the convolution of their prob. density function.

If A, B are random variables with $f_a(x)$, $f_b(x)$ $(a,b>0)$, calculate the prob. des. function of the random variable A+B.

<u>Solution</u>:

1) Need to check that $\int_{-\infty}^{\infty} f_a(x)dx = 1$.

Although it is straightforward by direct integration, we do it here using Fourier:

$$\mathcal{F}\left(e^{-a|x|}\right)(\omega) = \frac{2a}{a^2+\omega^2} \rightsquigarrow \mathcal{F}\left(\frac{2a}{a^2+\cancel{x}^2}\right)(\omega) = 2\pi\, e^{-a|\omega|}, \text{ so}$$

$$\mathcal{F}\left(\frac{a}{\pi(x^2+a^2)}\right)(\omega) = e^{-a|\omega|} \equiv \hat{f}_a(\omega)$$

Finally, $\int_{-\infty}^{\infty} f_a(x)\,dx = \mathcal{F}(f_a(x))(0) \equiv \hat{f}_a(0) = 1$ //.

2) By the definition given, if we denote $g_{a,b}(x)$ the prob. dens. function of A+B, we have that

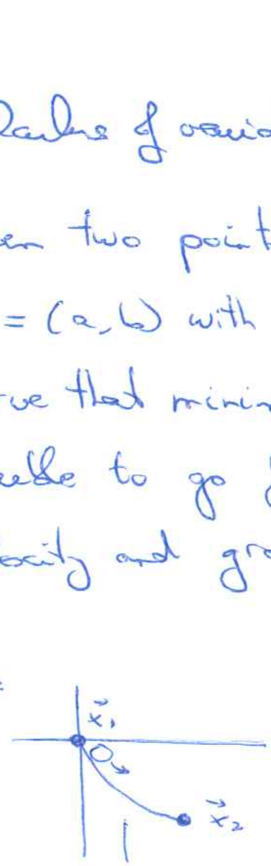
$$g_{a,b}(x) = (f_a * f_b)(x) = \mathcal{F}^{-1}(\hat{f}_a \hat{f}_b)(x) = \mathcal{F}^{-1}\left(e^{-a|\omega|} e^{-b|\omega|}\right)(x) =$$

$$= \mathcal{F}^{-1}\left(e^{-(a+b)|\omega|}\right)(x) = f_{a+b}(x) = \frac{a+b}{\pi\left(x^2 + (a+b)^2\right)}$$

PB | Calculus of variations: The Brachistochrone Problem.

Given two points in the plane $\vec{x}_1 = (0,0)$ and $\vec{x}_2 = (a,b)$ with $a > 0$, $b < 0$, find the shape of the curve that minimizes the time it would take a small marble to go from $\vec{x}_1$ to $\vec{x}_2$, starting with zero velocity and gravity as the only driving force.

Solution:



$y = z(x)$

Hint: The potential energy $(mg\,z(x))$ plus kinetic energy $\left(\frac{1}{2} m\, v(x)^2\right)$ has to remained constant.

Hint: $\int \sqrt{\frac{kz}{1-kz}}\,dz \sim$ Change $z = \frac{1}{2c}(1-\cos\theta)$

Total time from $\vec{x}_1$ to $\vec{x}_2$: $v = \frac{ds}{dt} \rightarrow T[z] = \int_0^L \frac{ds}{v(s)} = \int_0^a \frac{\sqrt{1+(z'(x))^2}}{v(x)}\,dx$

We need to write $v(x)$ in terms of $z(x)$:

$$\cancel{m}\, g\, z(x) + \frac{1}{2}\cancel{m}\,(v(x))^2 \underset{(\forall x \in [0,a])}{=} 0 \Rightarrow v(x) = \sqrt{2g(-z(x))}.$$

thus we need to minimize

$$\left.\begin{aligned} &E[z] = \int_0^a \sqrt{\frac{1+(z'(x))^2}{-2g\,z(x)}}\,dx \\ &\text{s.t. } z(0)=0,\ z(a)=b \end{aligned}\right\}$$

• Lagrangian: $F(x,z,p) = \sqrt{\dfrac{1+p^2}{-2gz}}$,

$$\frac{\partial F}{\partial p} = \frac{p}{\sqrt{-2gz(1+p^2)}},\quad \frac{\partial F}{\partial z} = -\frac{\sqrt{1+p^2}}{2\sqrt{2g}(-z)^{3/2}}$$, so we obtain the

Euler-Lagrange equation: $\left(\dfrac{\partial F}{\partial z} - \dfrac{d}{dx}\dfrac{\partial F}{\partial p} = 0\right)$

$$-\frac{\sqrt{1+(z')^2}}{2(-z)^{3/2}\sqrt{2g}} - \frac{d}{dx}\left(\frac{z'}{\sqrt{-2gz(1+(z')^2)}}\right) = -\frac{1+(z')^2+2zz''}{2\sqrt{2g}(-z)^{3/2}(1+(z')^2)^{3/2}} = 0 \Rightarrow$$

$$\left\| -\frac{1}{2}\frac{1+(z')^2+2zz''}{\sqrt{2g}\left(-z(1+(z')^2)\right)^{3/2}} = 0 \right\|$$ 2nd order ODE → Multiply by $z'(x)$ to find a total derivative

$$-\frac{1}{2}\,\frac{z' + (z')^3 + 2zz'z''}{\sqrt{2g}\,\left(-z\,(1+(z')^2)\right)^{3/2}} = \frac{d}{dx}\left(\frac{1}{\sqrt{2g}\,\sqrt{-z\,(1+(z')^2)}}\right) = 0 \Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{2g}\,\sqrt{-z(x)\left(1+(z'(x))^2\right)}} = c \quad \text{(1st order ODE: first integral of Euler-Lagrange equation).}$$

$$\downarrow$$

$$-z\,(1+(z')^2) = \frac{1}{2gc^2} \Rightarrow z'(x) = \sqrt{\frac{1+2gc^2 z(x)}{-2gc^2 z(x)}} \Rightarrow$$

$$\Rightarrow \int \sqrt{\frac{2gc^2(-z)}{1-2gc^2(-z)}}\, dz = x + d .$$

- Using the hint, let $-z = \frac{1}{2\cdot 2gc^2}(1-\cos\theta)$, so that

$$\frac{-1}{2\cdot 2gc^2}\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,\sin\theta\, d\theta = x + d$$

$$\parallel$$

$$\frac{-1}{2\cdot 2gc^2}\int \sqrt{\frac{1-\cos\theta}{1+\cancel{\cos\theta}}}\,\sqrt{(1\cancel{+\cos\theta})(1-\cos\theta)}\, d\theta = \frac{-1}{2\cdot 2gc^2}(\theta - \sin\theta) = x + d .$$

That is,
$$\left.\begin{aligned} -z(\theta) &= \frac{1}{2\cdot 2gc^2}(1-\cos\theta) \\ x(\theta) &= \frac{-1}{2\cdot 2gc^2}(\theta - \sin\theta) - d \end{aligned}\right\} \text{ with } c, d \text{ determined by the two points } \vec{x}_1, \vec{x}_2 .$$

- $\theta = 0 \Rightarrow z(0) = 0,\ x(0) = -d = 0 \Rightarrow \underline{\underline{d = 0}}$

- $a = x(\tilde{\theta}) = \frac{-1}{2\cdot 2gc^2}(\tilde{\theta} - \sin\tilde{\theta}),\ b = z(\tilde{\theta}) = \frac{1}{2\cdot 2gc^2}(1-\cos\tilde{\theta}) \longrightarrow$

The solution is the curve $(x(\theta), z(\theta))$, where

$$\left.\begin{array}{l} -z(\theta) = k\,(1-\cos\theta) \\ x(\theta) = -k\,(\theta - \sin\theta) \end{array}\right\} , \quad \theta \in [0, \tilde{\theta}], \text{ with}$$

$$\begin{cases} a = k\,(\tilde{\theta} - \sin\tilde{\theta}) \\ b = k\,(1-\cos\tilde{\theta}) \end{cases} \qquad (k, \tilde{\theta} \text{ determined by the data } a, b).$$

$\boxed{\text{P4}}$ Consider the annulus, $1 < x^2+y^2 < 2$ in $\mathbb{R}^2$, and the functional (A = the annulus)

$$E[u] = \iint_A \left(\frac{1}{2}|\nabla u|^2 - f\,u \right) dx\,dy, \text{ with domain the}$$

functions u such that $\left.\begin{array}{l} u(1,\theta) = \sin(2\theta) \\ u(2,\theta) = 8\sin(3\theta) \end{array}\right\}$ (conditions on ∂A in polar coordinates).

1) Write the weak form of the equation that any critical point of E must satisfy.

2) Write the strong form of the Euler-Lagrange equation for E.

3) Show that (among smooth functions), there is only one critical point of E.

4) Find the critical point of E when $f \equiv 0$.

1) We have to set the first variation of E to zero:

$$\langle \delta E[u], v\rangle = \frac{d}{d\varepsilon} E[u+\varepsilon v]\Big|_{\varepsilon=0} =$$

$$= \frac{d}{d\varepsilon}\iint_A \left(\frac{1}{2}\left((u_x+\varepsilon v_x)^2+(u_y+\varepsilon v_y)^2\right) - fu - f\varepsilon v\right) dx\,dy \,\Big|_{\varepsilon=0} =$$

$$= \iint_A \left((u_x+\varepsilon v_x)v_x + (u_y+\varepsilon v_y)v_y - fv\right) dx\,dy \,\Big|_{\varepsilon=0} =$$

$$= \iint_A (u_x v_x + u_y v_y - fv)\, dx\,dy .$$

- Weak formulation of E-L:

$$\begin{cases} \displaystyle\iint_A (u_x v_x + u_y v_y - fv)\, dx\,dy = 0 \quad \text{for all } v \text{ such that } v\big|_{\partial A} \equiv 0 . \\ u \text{ satisfying BC.} \end{cases}$$

2) Integrating by parts,

$$\langle \delta E[u], v\rangle = -\iint_A (u_{xx} v + u_{yy} v)\, dx\,dy + \underbrace{\int_{\partial A} u_x v n_1\, ds}_{=0} + \underbrace{\int_{\partial A} u_y v n_2\, ds}_{=0} +$$

$$-\iint_A fv\, dx\,dy =$$

$$= -\iint_A (u_{xx} + u_{yy} + f) v\, dx\,dy = 0 \implies$$

$$\Rightarrow \quad \left.\begin{array}{l} u_{xx} + u_{yy} = -f \quad \text{in } A \\ \left.\begin{array}{l} u(1,\theta) = \sin(2\theta) \\ u(2,\theta) = 8\sin(3\theta) \end{array}\right\} (\partial A) \end{array}\right\} (*)$$

3) Among smooth functions, critical points of E are the solutions of $(*)$. Let's show uniqueness for this PDE (Poisson eq.):

Let u, v be two solutions. Then, define $w = u - v$. It would satisfy that

$$\left.\begin{array}{l} \Delta w = w_{xx} + w_{yy} = 0 \quad \text{in } A \\ w = 0 \quad \text{on } \partial A \end{array}\right\} [W]$$

Multiply by w and integrate by parts:

$$\int_A \Delta w \, w \, dx\,dy = 0 \underset{\substack{\uparrow \\ \left(\substack{w=0 \\ \text{on } \partial A}\right)}}{=} -\int_A |\nabla w|^2 \, dx\,dy \underset{\substack{\uparrow \\ w \text{ is} \\ \text{smooth}}}{\Rightarrow} |\nabla w| = 0 \Rightarrow \nabla w = 0 \underset{\substack{\uparrow \\ (A \text{ is connected})}}{\Rightarrow}$$

$$\Rightarrow w \equiv \text{constant} \underset{\substack{\uparrow \\ \left(\substack{w = 0 \text{ on } \partial A \\ \text{and smooth}}\right)}}{\Rightarrow} w = 0 \Rightarrow u \equiv v \quad //.$$

[<u>Remark</u>: One could also proceed after [W] by applying the max. and min. principle for Laplace equation (A is connected, bounded, open)(w smooth).]

4) We have to solve (*) with $f \equiv 0$:

$$\begin{cases} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad (1<r<2,\ 0 \le \theta < 2\pi) \\ u(1,\theta) = \sin(2\theta) \\ u(2,\theta) = 8 \sin(3\theta) \end{cases}$$

Separation of variables: $u(r,\theta) = R(r)\varphi(\theta)$,

$$R''\varphi + \frac{1}{r}R'\varphi + \frac{1}{r^2}\varphi'' = 0 \to \left| \begin{array}{l} \varphi'' + \lambda\varphi = 0 \quad ① \\ r^2 R'' + r R' - \lambda R = 0 \quad ② \end{array} \right.$$

① $\left.\begin{array}{l} \varphi'' + \lambda \varphi = 0 \\ \text{periodic BC} \end{array}\right\} \Rightarrow \lambda = n^2,\ \varphi_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$

② $r^2 R'' + r R' - n^2 R = 0 \to R(r) = r^a \Rightarrow a = \pm n$, thus

$$\begin{cases} R_n(r) = c_n r^n + d_n \frac{1}{r^n} \quad (n \ge 1) \\ R_0(r) = c_0 + d_0 \log(r) \end{cases}$$

- So the solution is:

$$u(r,\theta) = a_0 + b_0 \log(r) + \sum_{n\ge1} r^n \left(a_n \cos(n\theta) + c_n \sin(n\theta)\right) +$$

$$+ \sum_{n\ge1} \frac{1}{r^n}\left(b_n \cos(n\theta) + d_n \sin(n\theta)\right).$$

• Finally, BC:

(1) $u(1,\theta) = a_0 + b_0 \log(1) + \sum_{n\geq 1} \Big(\cos(n\theta)(a_n+b_n) + (c_n+d_n)\sin(n\theta)\Big) = \sin(2\theta) \Rightarrow$

$\Rightarrow \; c_2 + d_2 = 1$

$a_n + b_n = 0$ for $n \geq 1$, $a_0 + b_0 \log(1) = 0$

$c_n + d_n = 0$ for $n \neq 2$

(2) $u(2,\theta) = a_0 + b_0 \log(2) + \sum_{n\geq 1} 2^n \big(a_n \cos(n\theta) + c_n \sin(n\theta)\big) +$

$+ \sum_{n\geq 1} \frac{1}{2^n}\Big(b_n \cos(n\theta) + d_n \sin(n\theta)\Big) = 8\sin(3\theta) \Rightarrow$

$\Rightarrow \; a_0 + b_0 \log(2) = 0$

$2^n a_n + \frac{b_n}{2^n} = 0$ for $n \geq 1$

$2^n c_n + \frac{d_n}{2^n} = 8$ for $n = 3$

$2^n c_n + \frac{d_n}{2^n} = 0$ for $n \neq 3$

Thus, $a_n = b_n = 0 \;\; \forall n \geq 0$

$c_n = d_n = 0 \;\; \forall n \neq 2, 3$, and

$\left.\begin{array}{l} c_2 + d_2 = 1 \\ 2^2 c_2 + \frac{d_2}{2^2} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_2 = \dots \\ d_2 = \dots \end{array}$

$\left.\begin{array}{l} 2^3 c_3 + \frac{d_3}{2^3} = 8 \\ c_3 + d_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_3 = \dots \\ d_3 = \dots \end{array}$

and final solution is

$$u(r,\theta) = c_2 r^2 \sin(2\theta) + c_3 r^3 \sin(3\theta) + $$
$$+ \frac{d_2}{r^2} \sin(2\theta) + \frac{d_3}{r^3} \sin(3\theta).$$