Name: ______________________________ **PennID**: ______________

Math 425/AMCS 525
Final exam (Practice exam)
9:00-11:00, May 13, 2019

Please *turn off and put away all electronic devices.* You are allowed to use on side of an 8x11 cheat-sheet with hand-written notes. No calculators, no books. Read the problems carefully. **Show all work** (answers without proper justification will not receive full credit). Be as organized as possible: illegible work will not be graded.

Please sign and date the pledge below to comply with the Code of Academic Integrity. Don't forget to write your Name and PennID on the top of this page. Good luck!

#	Points possible	Your score
1	15	
2	20	
3	15	
4	12.5	
5	12.5	
6	12.5	
7	12.5	
Total	100	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

______________________________ ______________

Signature Date

Fourier Transform

	$f(x)$	$F(k)$
Delta function	$\delta(x)$	1
Square pulse	$H(a-\|x\|)$	$\dfrac{2}{k}\sin ak$
Exponential	$e^{-a\|x\|}$	$\dfrac{2a}{a^2+k^2} \quad (a>0)$
Heaviside function	$H(x)$	$\pi\delta(k)+\dfrac{1}{ik}$
Sign	$H(x)-H(-x)$	$\dfrac{2}{ik}$
Constant	1	$2\pi\delta(k)$
Gaussian	$e^{-x^2/2}$	$\sqrt{2\pi}\,e^{-k^2/2}$

	Function	**Transform**
(i)	$\dfrac{df}{dx}$	$ikF(k)$
(ii)	$xf(x)$	$i\dfrac{dF}{dk}$
(iii)	$f(x-a)$	$e^{-iak}F(k)$
(iv)	$e^{iax}f(x)$	$F(k-a)$
(v)	$af(x)+bg(x)$	$aF(k)+bG(k)$
(vi)	$f(ax)$	$\dfrac{1}{\|a\|}F\left(\dfrac{k}{a}\right) \quad (a\neq 0)$

Problem 1 (15 pts)

Part a. Use the coordinate method to find the function $u(x,t)$ that solves:

$$u_{xx} - 3u_{xt} + 2u_{tt} = 2x + 2t.$$

Part b. Solve $\sqrt{1-x^2}u_x + u_y = 0$ with $u(0,y) = y$.

Solution (Problem 1):

Problem 2 (20 pts)

Part a. Solve $u_t = u_{xx}$, $u(x,0) = e^{-x}$, $u(0,t) = 0$ on the half-line $0 < x < \infty$.

Part b. A rod has length $L = 1$ and constant $k = 1$. Its temperature satisfies the heat equation. Its left end is held at temperature 0 and *its right end at temperature* 1. Initially the temperature is given by

$$\phi(x) = \begin{cases} \dfrac{5x}{2}, & 0 \leq x \leq \dfrac{2}{3}, \\ 3 - 2x, & \dfrac{2}{3} < x \leq 1 \end{cases}$$

Find the solution as a series (compute the coeffients).

Part c. Does the sine Fourier series of $\phi(x)$ converges pointwise to $\phi(x)$ in $0 \leq x \leq 1$? Is the convergence uniform?

Solution (Problem 2):

Solution (Problem 2):

Problem 3 (15 pts)

Choose two:

Part a. (1-d Poincaré inequality) Show that if $f(x)$ is a C^1 function in $[-\pi, \pi]$ that satisfies periodic boundary conditions and if $\int_{-\pi}^{\pi} f(x)dx = 0$, then

$$\int_{\pi}^{\pi} |f(x)|^2 dx \leq \int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

Hint: Use Parseval's equality.

Part b. Show that if $f(x)$, $f'(x)$ are continuous periodic functions of period 2π, then its classical Fourier series converges to f uniformly on $\mathbb{R}$.

Part c. Consider the second order differential operator $A = -\frac{d^2}{dx^2}$ with symmetric boundary conditions. Show that if, in addition,

$$f(x)f'(x)\Big|_{x=a}^{x=b} \leq 0,$$

for any function $f(x)$ satisfying the boundary conditions, then there is no negative eigenvalue for A.

Solution (Problem 3):

Solution (Problem 3):

Problem 4 (12.5 pts)

Solve the Laplace equation

$$u_{xx} + u_{yy} = 0$$

in the annulus $\{1 < x^2 + y^2 < e^2\}$ subject to the following boundary conditions:

$$\begin{aligned} &u(x, y) = 0 \quad \text{on the inner circle,} \\ &u(x, y) = 5 + (e^4 - 1)\cos(2\theta) \quad \text{on the outer circle,} \end{aligned}$$

where θ is the angle of the point (x, y) measure from the origin.

Solution (Problem 4):

Solution (Problem 4):

Problem 5 (12.5 pts)

Part a. Let $u \geq 0$ and $\Delta u = 0$ in the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$. Using the Mean-Value Property for harmonic functions, prove the following version of the so-called Harnack inequality

$$\frac{1-r}{1+r}u(0,0) \leq u(x,y) \leq \frac{1+r}{1-r}u(0,0),$$

where $r = \sqrt{x^2 + y^2} < 1$.

Part b. Consider the following problem

$$\begin{aligned} &\Delta u = 0, \quad D = \{(x, y) : x^2 + y^2 \leq 1\}, \\ &u = h, \qquad \text{on } \partial D. \end{aligned}$$

Part b.1. Show that if $h \geq 0$ then $u > 0$ unless $h \equiv 0$.

Part b.2. Let $u(0) = 1$ and $h \geq 0$. Show that

$$\frac{1}{3} \leq u(x,y) \leq 3 \quad \text{ for all } x^2 + y^2 = \frac{1}{4}.$$

Solution (Problem 5):

Solution (Problem 5):

Problem 6 (12.5 pts)

Part a. Use the calculus of variations to prove that the shortest path between two given points is a straight line. That is, define an appropriate functional to minimize and find its critical points (do not need to show they are indeed a minimum).

Part b. Let D be the unit disk in $\mathbb{R}^2$. Consider the following functional

$$E[u] = \iint_D \frac{1}{2}|\nabla u|^2 dxdy$$

with domain the smooth functions such that $u \equiv 2$ on ∂D.

Part b.1. Compute the first variation of E and find the strong form of the Euler-Lagrange equation for it.

Part b.2. If u is a critical point of E, find $g(x,y) = u_x(x,y) - u(x,y)$.

Solution (Problem 6):

Solution (Problem 6):

Problem 7 (12.5 pts)

Part a. Let $f_a(x) = \dfrac{a}{\pi(x^2 + a^2)}$, $a > 0$.

Part a.1. Show that $f_a(x)$ is a probability density function, that is,

$$f_a(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f_a(x)dx = 1.$$

Part a.2. The probability density function of the sum of two random variables is the convolution of their probability density functions. If A, B are random variables with $f_a(x)$, $f_b(x)$ $(a, b > 0)$, calculate the probability density function of the random variable $A + B$. Give the results without integrals.

Part b. Consider the wave equation

$$\begin{aligned} &u_{tt} = u_{xx}, \quad -\infty < x < \infty, t > 0, \\ &u(x, 0) = g(x), \\ &u_t(x, 0) = h(x). \end{aligned}$$

Obtain D'Alembert formula using the Fourier transform.

Solution (Problem 7):

Solution (Problem 7):

Extra paper

Extra paper

Extra paper

Extra paper