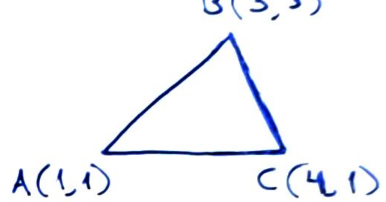
MATH 115 LECTURE 1 : Review of vectors (~ Sections 12.1 – 12.5)

The goal of today's lecture is to review the language of vectors in the plane and in space, which will be needed in multivariate calculus to talk about surfaces.

We want to answer questions such as:

1) What are the angles of the triangle



?

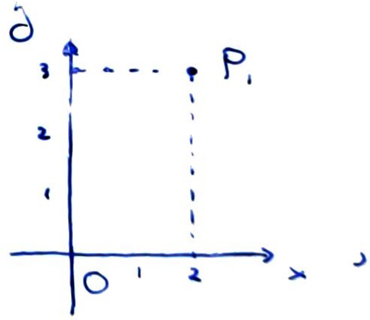
2) How can we describe a line or a plane?

3) How do we find the distance between a given point and a line or a plane?

12.1 Points and distances

• <u>2d</u>

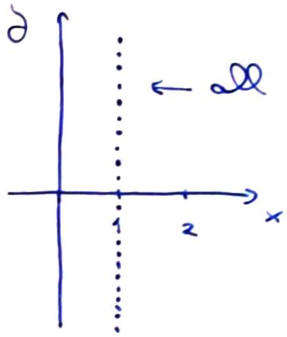
Once we fixed a coordinate system



, we describe any point by its <u>coordinates</u>. For example, we write $P_1(2,3)$, that is, P_1 is the point whose coordinates are $x = 2$, $y = 3$.

Question: Where do all points whose first coordinate is equal to 1 lie?

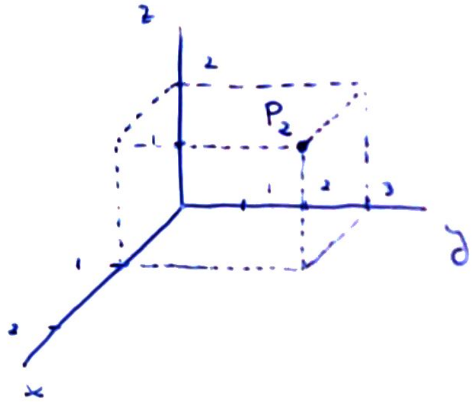
That is, what does $x = 1$ mean geometrically?



← all those points: a line.

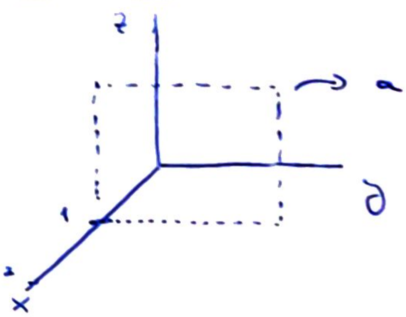
• 3d

We do exactly the same in the space: $P_2(1, 3, 2)$



Remark: The position of the axis is important. We always use a "right-handed" coordinate system.

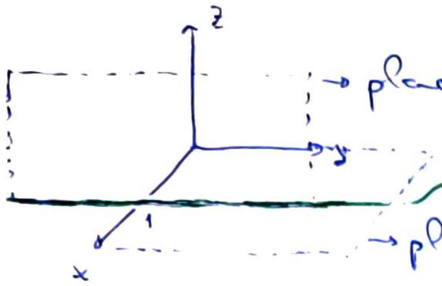
Question: • What is the geometric meaning of $x = 1$ now?



→ a plane

• And what about the equations $x=1$ and $z=0$?

↳ This is a <u>line</u> (indeed, the intersection of the planes $x=1$ and $z=0$)



• <u>Important</u>: How do we compute the distance between two points?

<u>Exercise</u>: Find the distance between the two given points.

a) $P_1(0,0)$, $P_2(0,1)$

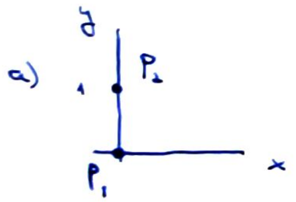
b) $P_1(0,0)$, $P_2(2,1)$

c) $P_1(1,2)$, $P_2(2,4)$

d) $P_1(0,0,0)$, $P_2(1,2,1)$

Let's draw each case:

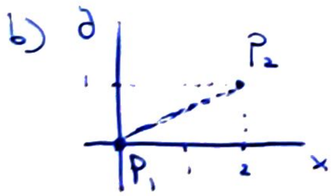
a)



Clearly, $|P_1P_2| = 1$

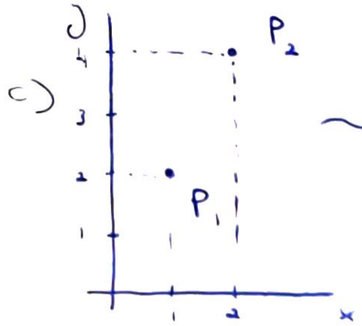
that's how we denote "distance between P_1 and P_2".

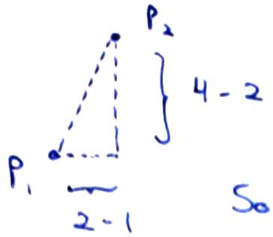
b)



We use the Pythagorean theorem.

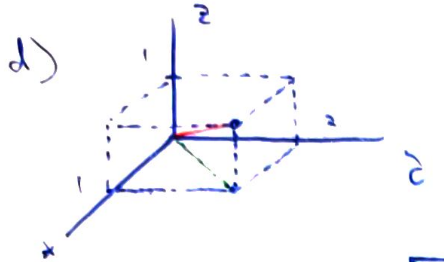
$$|P_1P_2|^2 = 1^2 + 2^2 \rightarrow |P_1P_2| = \sqrt{1^2+2^2} = \sqrt{5}.$$

c) 

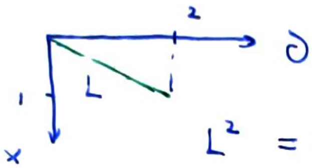


So by Pythagoras,

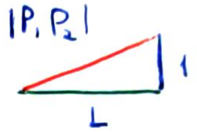
$$|P_1P_2|^2 = 1^2 + 2^2 \rightarrow |P_1P_2| = \sqrt{5}.$$

d) 

We use Pythagoras twice:



$$L^2 = 1^2 + 2^2 \longrightarrow$$



Thus,

$$|P_1P_2|^2 = L^2 + 1^2 = 1^2 + 2^2 + 1^2.$$

$$\Rightarrow |P_1P_2| = \sqrt{1^2 + 2^2 + 1^2}.$$

• **Formula**: Distance between two points.

Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, the distance between them is

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}$$

Exercise: An sphere centered at $P_0(x_0, y_0, z_0)$ of radius R is defined as the set of points that lie at distance R from P_0.

Find the equation that any point $P(x,y,z)$ on the sphere must satisfy.

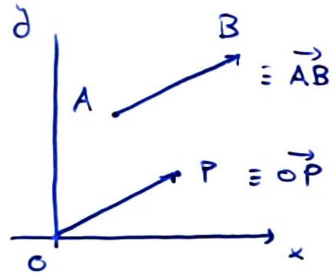
Solution:

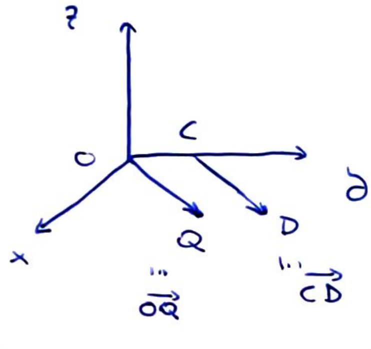
$$|PP_0| = a \iff \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = a \iff$$

$$\iff \quad (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2$$

12.2 | Vectors

We can think of vectors as "directed line segments":





They can represent velocities or forces, and are defined by its length and direction: Two vectors are equal if they have same length and same direction.

→ For example, $\vec{AB} = \vec{OP}$, $\vec{OQ} = \vec{CD}$.

• Given two points A and B, the vector $\vec{AB}$ is perfectly defined:

- Its length (or magnitude or modulus) is $|\vec{AB}| = |AB|$

- Its direction is given by the directed segment from A to B.

<u>Important</u>: Since a vector does not depend on the points (recall the picture where $\vec{AB} = \vec{OP}$), we want a better way of working with them (moreover, we want to use numbers, not arrows!).

• <u>Definition</u>:

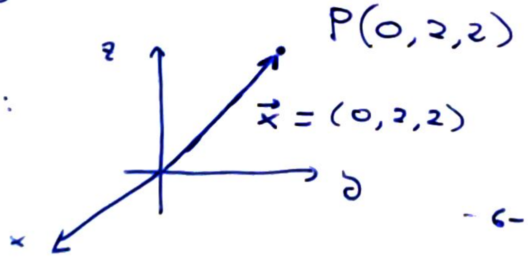
1) A two-dimensional vector $\vec{v}$ is an ordered pair of real numbers $\vec{v} = (v_1, v_2)$, where $v_1, v_2 \in \mathbb{R}$ are called the <u>components</u> of $\vec{v}$.

2) Analogously for three-dimensional vectors $\vec{v} = (v_1, v_2, v_3)$.

→ <u>Relationship with points</u>:

We can identify any point $P(x, y, z)$ with a vector $\vec{x} = (x, y, z)$.

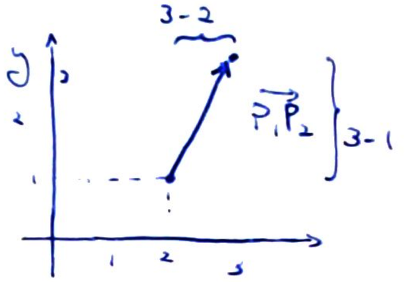
Example:



Notice that we are identifying $P(x,y,z)$ with the oriented segment $\vec{OP}$.

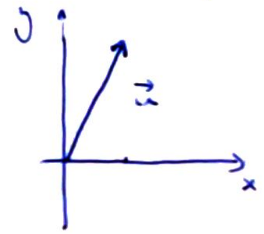
• Exercise: Find the components of the vector $\vec{P_1P_2}$, where $P_1(2,1)$, $P_2(3,3)$. Find $|\vec{P_1P_2}|$.

Sol:



Recall that $\vec{P_1P_2}$ is equal to any "arrow" with same length and direction, so in particular equal to

$|\vec{P_1P_2}| = |\vec{u}| =$
$= \sqrt{u_1^2+u_2^2} = \sqrt{5}$.



$\vec{u} = (3-2,\ 3-1) = (1,2)$

• General: Give two points $P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$, the vector $\vec{PQ}$ is $\vec{PQ} = \vec{v} = (x_2-x_1,\ y_2-y_1,\ z_2-z_1)$.

Its length is $|\vec{PQ}| = |\vec{v}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

→ Review at home: operations with vectors.

- Addition: $\vec{u}+\vec{v}=(u_1+v_1, u_2+v_2)$.
- Scaling: $k\vec{u}=(ku_1, ku_2)$, where k is a <u>scalar</u> (a number).

- <u>Unit vectors</u>

We have now a way to represent vectors analitically, through their components. But we must not forget what they represent: a directed segment.

That is, we are dealing with some "object moving in a certain direction with a certain speed" or some "force applied in a direction with a certain strength".

So sometimes we are interested in writing vectors as

"magnitude times direction".

- <u>Def</u>: Unit vector.

We called unit vector to those vectors with magnitude equal to 1.

$\vec{u}$ is unit vector if $|\vec{u}|=1$.

- The standard unit vectors are $\left.\begin{array}{l}\vec{i} = (1,0,0) \\ \vec{j} = (0,1,0) \\ \vec{k} = (0,0,1)\end{array}\right\}$

<u>Remark</u>: We can write any vector as a linear combination of the standard unit vectors,

$\vec{u} = (u_1, u_2, u_3) = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$.

<u>Exercise</u>: Write $\vec{AB}$ in the form $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$, if

$A(-7,-8,1)$, $B(-10,8,1)$.

<u>Sol</u>:

$$\vec{AB} = \vec{v} = (-10-(-7),\ 8-(-8),\ 1-1) = (-3, 16, 0) =$$
$$= -3\vec{i} + 16\vec{j} + 0\vec{k}.$$

- <u>Definition</u>: The "direction vector" of a vector $\vec{u}$ (usually called simple the direction of $\vec{u}$) is the <u>unit vector</u> with the same direction than $\vec{u}$.

Exercise: Express $\vec{a}$ as a product of its length and direction.

$$\vec{a} = 2\vec{i} + \vec{j} - 2\vec{k}.$$

Sol:

Length $|\vec{a}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3.$

So $\vec{a} = \frac{|\vec{a}|}{|\vec{a}|}(2\vec{i} + \vec{j} - 2\vec{k}) = 3\,\frac{2\vec{i} + \vec{j} - 2\vec{k}}{3}$

Note that $\frac{2\vec{i} + \vec{j} - 2\vec{k}}{3}$ is a unit vector!

$\hookrightarrow \left|\frac{2\vec{i} + \vec{j} - 2\vec{k}}{3}\right| = \sqrt{\frac{2^2}{3^2} + \frac{1^2}{3^2} + \frac{2^2}{3^2}} = \frac{1}{3}\cdot 3 = 1.$

So,

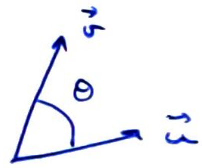
$$\vec{a} = \underbrace{3}_{\text{length}}\underbrace{\left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)}_{\text{direction}}.$$

→ In general: $\vec{u} = \underbrace{|\vec{u}|}_{\text{length}}\underbrace{\frac{\vec{u}}{|\vec{u}|}}_{\text{direction}}$

(if $\vec{u} \neq \vec{0}$)

12.3 The dot product: angles and projections.

Any two given vectors define an angle



So now we want to use vectors to compute angles easily.

- **Definition:** Dot product

The dot product $\vec{u}\cdot\vec{v}$ of vectors $\vec{u}$ and $\vec{v}$ is a number defined as

$$\boxed{\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos(\theta)}$$

, where θ is the angle between $\vec{u}$

Ok, this is not very useful now... we wanted to compute θ!

- **Theorem:** The dot product $\vec{u}\cdot\vec{v}$ of vectors

$\vec{u} = (u_1, u_2, u_3) = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and
$\vec{v} = (v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, is given by

$$\boxed{\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3}$$

[Proof: See book (uses law of cosines)]

- **Remark:** Vectors $\vec{u}, \vec{v}$ are orthogonal (i.e., perpendicular) if and only if $\boxed{\vec{u}\cdot\vec{v} = 0}$ Orthogonal vectors $\left[\cos\left(\frac{\pi}{2}\right) = 0\right]$.

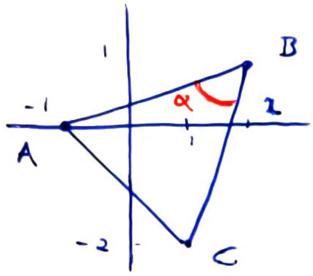
• Conclusion: Given two vectors $\vec{u}, \vec{v}$, the angle between them is given by

$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\frac{u_1v_1+u_2v_2+u_3v_3}{|\vec{u}||\vec{v}|}\right)$$

since $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$.

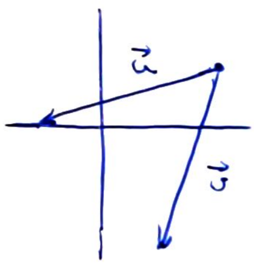
$\vec{u}\cdot\vec{v} = u_1v_1+u_2v_2+u_3v_3$.

• Exercise: Find the measures of the angles of the following triangle



A (-1,0)
B (2,1)
C (1,-2)

Sol: Let's do for example $\alpha = \widehat{ABC}$. We can do it using vectors:



$\vec{u} = \vec{BA} = (-1-2, 0-1) = (-3,-1)$

$\vec{v} = \vec{BC} = (1-2, -2-1) = (-1,-3)$

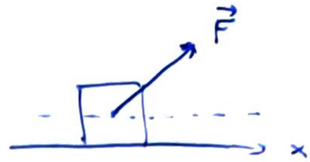
$|\vec{u}| = \sqrt{9+1} = \sqrt{10} = |\vec{v}|$

Therefore,

$$\alpha = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \arccos\left(\frac{6}{10}\right).$$

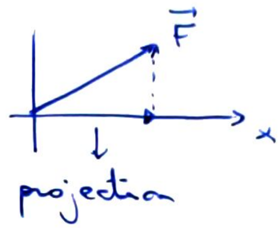
Dot products are also very useful to compute "projections".

Say we pull a box as follows



If the box is heavy enough, it will only move in the direction of the x axis.

But the net force in this direction is not $\vec{F}$, but only its projection onto the x axis



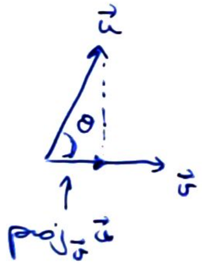
<u>Def</u>: Projection of a vector $\vec{u}$ onto $\vec{v}$

$$\boxed{\mathrm{proj}_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}}$$

Length: $|\vec{u}|\cos\theta$

Direction: $\dfrac{\vec{v}}{|\vec{v}|}$

<u>Remark</u>: This corresponds to our intuition



$$|\mathrm{proj}_{\vec{v}}\,\vec{u}| = \frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|} = |\vec{u}|\cos\theta \quad \checkmark$$

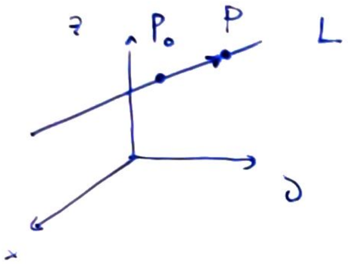
12.5 Lines and planes in space

1) Lines

- Parametric equations for a line

Consider a line L that goes through $P_0(x_0, y_0, z_0)$ and is parallel to a vector $\vec{v} = (v_1, v_2, v_3)$.

How can we describe all the points $P(x, y, z)$ that lie in L?



Well, for any $P(x, y, z)$ on the line, the vector $\overrightarrow{P_0P}$ has to be parallel to $\vec{v}$.

Recall that $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$, and two vectors are parallel only if they are multiples of each other. Therefore it must be that

$\overrightarrow{P_0P} = t\vec{v}$, where $t \in \mathbb{R}$ is the so-called parameter.

$\hookrightarrow (x - x_0, y - y_0, z - z_0) = (tv_1, tv_2, tv_3) \Rightarrow$

$$\Rightarrow \left.\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}\right\}$$

Parametric equations of L

Line through $P_0(x_0, y_0, z_0)$, parallel to $\vec{v} = (v_1, v_2, v_3)$.

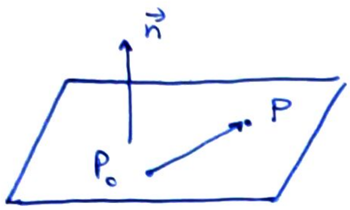
(for each $t \in \mathbb{R}$ we find a different point of L).

2) Planes

Consider a plane whose "normal" (vector perpendicular to the plane) is

$\vec{n} = (A, B, C)$, and that goes through $P_0(x_0, y_0, z_0)$.

Then, for any point $P(x, y, z)$ on the plane we must have that:



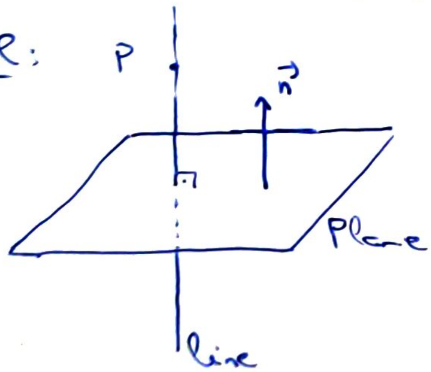
$\| \overrightarrow{P_0P}$ is perpendicular to $\vec{n}$ $\|$

That is, $\overrightarrow{P_0P} \cdot \vec{n} = 0 \rightsquigarrow (x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0 \rightsquigarrow$

$$\rightsquigarrow \| \; A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \; \Big|$$ Equation for plane through (x_0, y_0, z_0) with normal (A, B, C).

<u>Exercise</u>: Find the parametric equations for the line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$.

<u>Sol</u>:



$\begin{cases} P(2,4,5) \\ \text{Plane } 3x + 7y - 5z = 21 \rightarrow \vec{n} = (3, 7, -5) \end{cases}$

So we want line through P parallel to $\vec{n}$:

$P(x, y, z)$ is on the line if $\vec{x} = (2, 4, 5) + t(3, 7, -5)$,

that is, the parametric equations of the line are

$$\left.\begin{aligned} x &= 2+3t \\ y &= 4+7t \\ z &= 5-5t \end{aligned}\right\} t\in\mathbb{R}$$

<u>Exercise</u>: Find the equation for the plane through $A(1,-2,1)$ perpendicular to the vector from the origin to A.

<u>Sol</u>: The vector $\vec{OA} = (1,-2,1)$ is perpendicular to the plane, that is, it's its normal $\vec{n} = (1,-2,1)$.

So plane through $A(1,-2,1)$ with $\vec{n} = (1,-2,1)$ is

$$(x-1) - 2(y+2) + (z-1) = 0.$$