MATH 115
LECTURE 2

: Functions of several variables.

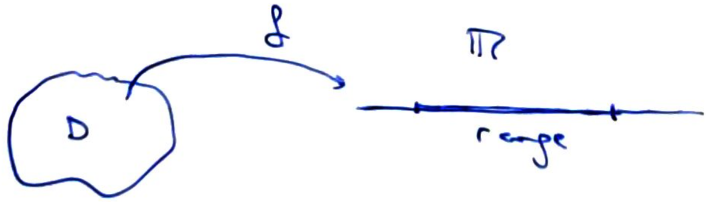
14.1 Functions of several variables.

Def: Let D be a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$.
A real function f on D is a rule that assigns a unique real number to each element in D,
denoted $\underbrace{f(x_1, x_2, \dots, x_n)}_{\text{real number}}$.

- The set D is called the domain of f.
- The variables $x_1, x_2, \dots, x_n$ are the independent variables.
  The output variable, that is, $f(x_1, \dots, x_n)$, is the dependent variable.
  ↳ In 2d we usually write $z = f(x,y)$.
- All the number that can be reached by f are called the range.

→ Recall 1d: $y = f(x)$ (domain, range)

Remark:



Domain $\subseteq \mathbb{R}^n$

Range $\subseteq \mathbb{R}$

$\mathbb{R}^n$
(several variables)

Example: Find the domain and range of the following functions:

1) $f(x,y) = x^2 + 2y^2$ $\qquad [\, f(1,1)?\ f(2,1)? \dots]$

Domain: All $\mathbb{R}^2$

Range: Notice that $\left.\begin{array}{l} x^2 \geq 0 \\ y^2 \geq 0 \end{array}\right\} \Rightarrow f(x,y) \geq 0 \Rightarrow$ Range is $[0, +\infty)$.

2) $f(x,y,z) = x - 2y + 3z$

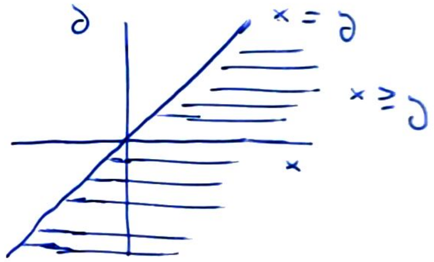
$D(f) = \mathbb{R}^3$, $R(f) = \mathbb{R}$

3) $f(x,y) = \cos(x)\sin(y)e^{xy} + \sqrt{x-y}$

- Domain: We need $x - y \geq 0$ for $f(x,y)$ to be defined. So

$$D(f) = \{(x,y) \in \mathbb{R}^2 : x - y \geq 0\}$$

That is, the domain is a half-plane

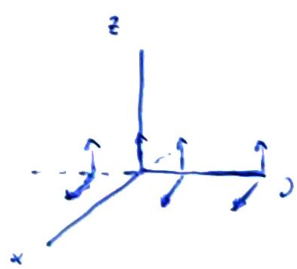


- Range: $\mathbb{R}$.

4) $f(x,y,z)$ $x\log(z) + y\log(x)$.

Domain: We need $z > 0$ and $x > 0$, so

$$D(f) = \{(x,y,z) \in \mathbb{R}^3 : x > 0,\ z > 0\}, \text{ that is}$$



Open, closed, bounded sets

Def: A point (x_0, y_0) in a region R of the x-y plane is

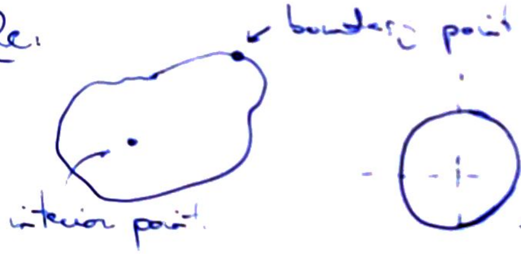
- an interior point of R if there exist a disk centered in (x_0, y_0) that lies entirely in R.

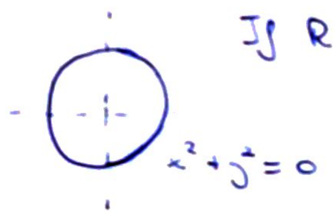
- a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie inside of R.

Def:
- Interior of R: set of all the interior points of R.
- Boundary of R: " " " " boundary " " ".

Def:
- A region is open if it is equal to its interior
- A region is closed if it contains all its boundary points.

Example:





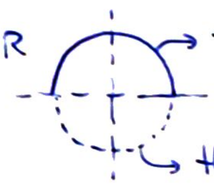
If $R = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

↳ Boundary: $x^2+y^2=1$

Interior: $x^2+y^2<1$.

- **Remark:** A set might be not open nor closed!

R



→ these are boundary points (so R it's not open).

→ these boundary points are not contained in R (so it isn't closed).

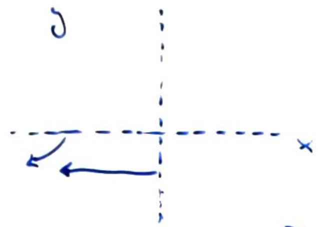
• Def: A set is bounded if it lies inside of a (sufficiently big) disk. Otherwise it is called unbounded.

Examples: Decide if the domain of f is open, closed, bounded, unbounded:

1) $f(x,y) = \dfrac{1}{xy}$

Domain: We need that $xy \neq 0 \to D(f) = \{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$.

That is,



these are boundary points.

• Unbounded.
• Not closed.
• Open.

2) $f(x,y) = \arccos(y - x^2)$ [Recall that $\cos\theta \in [-1,1]$].

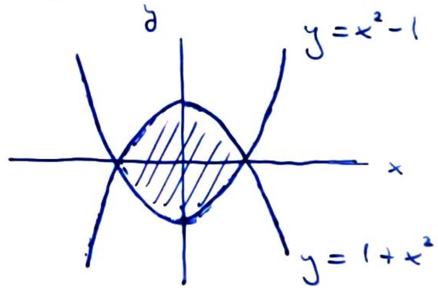
↳ For $\arccos(y-x^2)$ to be defined, we need $-1 \leq y - x^2 \leq 1$.
So,

$$D(f) = \{(x,y) \in \mathbb{R}^2 : -1 \leq y - x^2 \leq 1\}$$

We can make a sketch:

$-1 \leq y - x^2 \to y \geq x^2 - 1$

$y - x^2 \leq 1 \to y \leq 1 + x^2$



So the domain is | closed.
bounded.
(not closed).

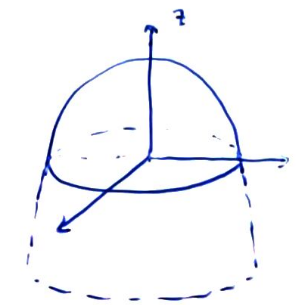
3) $f(x,y) = \log(1 - x^2 - y^2)$ ... (bounded, open)

• Remark: All the previous definitions are the same in 3d by simply saying ball instead of disk.

• || Graphs, level and contour curves, level surfaces

Def: The graph of $f(x,y)$ is the set of points in $\mathbb{R}^3$ given by $(x,y,f(x,y))$ where (x,y) live in the domain of f.

Ex: $f(x,y) = 1-x^2-y^2$.

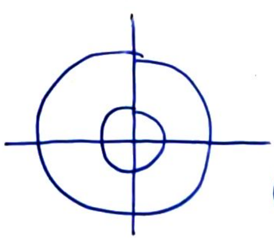


Def: A level curve of f is the set of points in the plane where $f(x,y)$ has a constant value

Ex: Level curves of $f(x,y) = 1-x^2-y^2$ are the curves

$$1-x^2-y^2 = c \quad \text{for } c \in \mathbb{R}$$

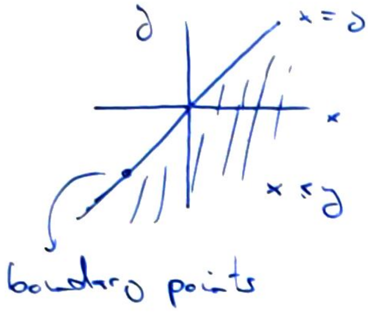
$\hookrightarrow$ Circles of radius $1-c$ centered at $(0,0)$



$\rightarrow$ Corresponds to cutting the graph of $f(x,y)$ with the plane $z=c$ (and projecting the result to the plane xy).

· Exercise: For $f(x,y) = \sqrt{y-x}$, find domain, range, level curves, boundary of the domain and decide if the domain is open, closed, bounded or unbounded.

Sol. $D(f) = \{(x,y) \in \mathbb{R}^2 : y - x \geq 0\}$. $R(f) = [0, +\infty)$



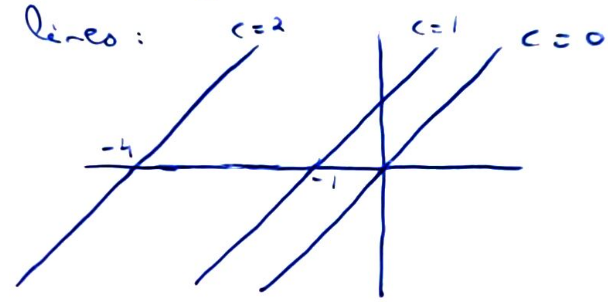
$D(f)$ is closed, unbounded (not open).

· Level curves:

$$f(x,y) = \sqrt{y-x} = c \Leftrightarrow$$

$$\Leftrightarrow y - x = c^2 \Leftrightarrow y = x + c^2$$

So the level curves are straight lines:



· Exercise: Find an equation and sketch the level curve of $f(x,y) = \sqrt{x^2-1}$ that passes through $(1,0)$.

Sol.

$$\sqrt{x^2-1} = c \underset{(1,0)}{\rightsquigarrow} c = 0 \rightsquigarrow \sqrt{x^2-1} = 0 \Leftrightarrow x = \pm 1.$$