MATH 115 LECTURE 3 : Limits and Continuity in Higher Dimensions.

(~ Section 14.2)

Continuation of Lecture 2.

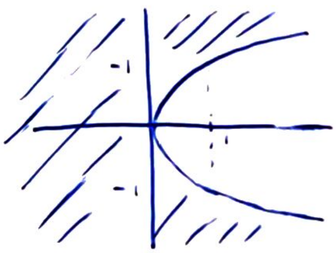
- → Graph
- → Level curves

($f(x,y) = 1 - x^2 - y^2$)

• **Exercise:** For $f(x,y) = \sqrt{y^2 - x}$, find domain, range, describe the level curves, find the boundary of the domain and determine if the domain is open/closed/bounded.

Sol: Domain $D(f) = \{(x,y) \in \mathbb{R}^2 : y^2 - x \geq 0\}$

↳ Sketch:



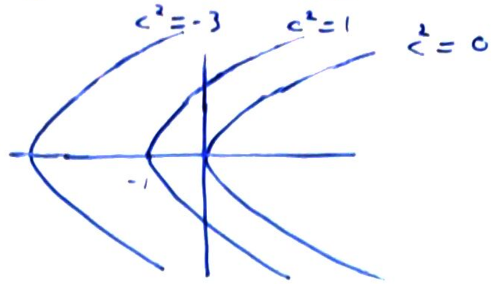
$y = x^2$ $\quad\Leftrightarrow x \le y^2$

- Boundary of $D(f) = \{(x,y) \in \mathbb{R}^2 : y^2 = x\}$
- All the boundary points are in the domain $\Rightarrow$ $D(f)$ is closed. (not open).
- $D(f)$ is unbounded (not contained in a disk).
- Range of f: $R(f) = [0, +\infty)$

• Level curves: $f(x,y) = \sqrt{y^2 - x} = c \rightarrow$ $(c \geq 0)$

→ Curves $\sqrt{y^2 - x} = c \Leftrightarrow y^2 - x = c^2 \Leftrightarrow \| y^2 = x + c^2 \|$.

These are parabolas:
with vertex $(-c^2, 0)$.



• Exercise: Match pictures...

• Def: Level surface of $f(x,y,z)$

↳ It is a surface in $\mathbb{R}^3$ where the function has constant values.

• Exercise: Find an equation for the level surface of the function $f(x,y,z) = \ln(x^2 + y^2 + z^2)$ through the point $(-1, 2, 1)$.

Sol:

$$\ln(x^2 + y^2 + z^2) = c \Leftrightarrow x^2 + y^2 + z^2 = e^c$$

So the level surfaces of f are spheres centered at the origin with radius $\sqrt{e^c}$.

In particular, $1 + 4 + 1 = e^c = 6 \rightsquigarrow x^2 + y^2 + z^2 = 6$ is the one through $(-1, 2, 1)$.

- **Exercise:** Find and sketch the domain of $f(x,y)=\sum_{n=0}^{\infty}\left(\frac{x}{y}\right)^n$. Then find an equation for the level curve that passes through $(1,2)$.

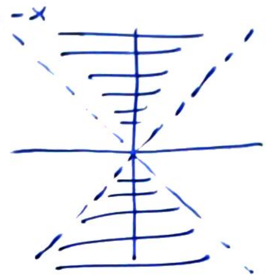
Sol:

Domain: The geometric infinite series is defined only for $\left|\frac{x}{y}\right|<1$, so $D(f)=\{(x,y)\in\mathbb{R}^2 : \left|\frac{x}{y}\right|<1\}$.

That is,

$$D(f)=\{(x,y)\in\mathbb{R}^2 : -1<\frac{x}{y}<1\}=$$

$$=\{(x,y)\in\mathbb{R}^2 : y>0,\ -y<x<y\}\cup\{(x,y)\in\mathbb{R}^2 : y<0,\ -y>x>y\}$$



Now, level curves $\sum_{n=0}^{\infty}\left(\frac{x}{y}\right)^n=c$.
The one through $(1,2)$ has $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=c$ $\Big\} \rightarrow$ So the level curve that goes through $(1,2)$ is

$$\sum_{n=0}^{\infty}\left(\frac{x}{y}\right)^n=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \quad \Big\|.$$

to obtain full grade

Remark: We can simplify this,

$$\left.\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{x}{y}\right)^n&=\frac{1}{1-x/y}=\frac{y}{y-x}\\ \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n&=\frac{1}{1-1/2}=2\end{aligned}\right\}\rightarrow \frac{y}{y-x}=2 \Leftrightarrow \boxed{y=2x}$$

14.2 Limits and Continuity

• Def: Limit

We say that the limit of $f(x,y)$ at (x_0,y_0) is L, and denoted that by

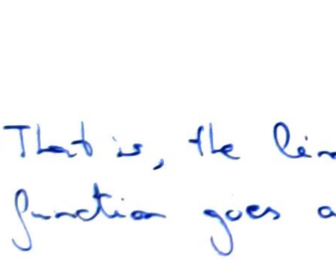
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

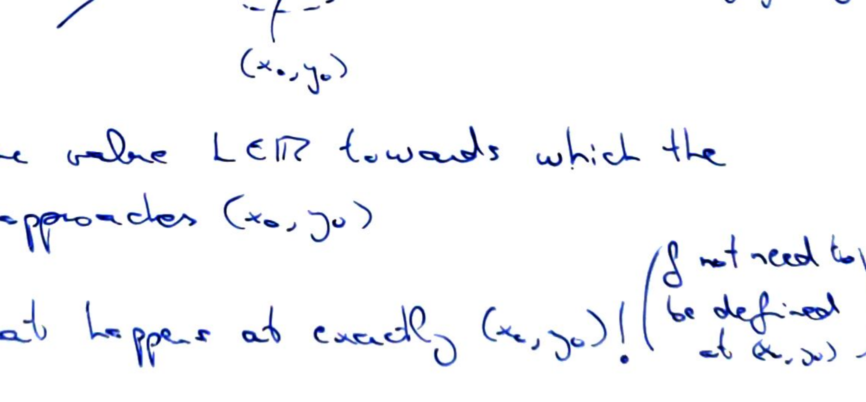
if

$$\forall \varepsilon > 0,\ \exists \delta = \delta(\varepsilon) > 0 \ / \ \left\{ \begin{array}{l} 0 < |(x,y)-(x_0,y_0)| < \delta \\ (x,y) \in D(f) \end{array} \right\} \Rightarrow |f(x,y) - L| < \varepsilon.$$

(for all)

Remark: It is the same definition as in 1-d, but we must consider all points (x,y) in a neighbourhood of (x_0,y_0): in 1-d that was an interval, now a disk.





• That is, the limit is the value $L \in \mathbb{R}$ towards which the function goes as (x,y) approaches (x_0,y_0)

↳ It does not matter what happens at exactly (x_0,y_0)! (f not need to be defined at (x_0,y_0)).

↳ In 2-d, (x,y) can approach (x_0,y_0) from many directions!

"Methods" for exercises

1) Direct substitution + standard rules (assuming knowledge of 1-d limits).

Summary: The limit of the sum, product, quotient, composition of functions is the sum, product, quotient of the limits (if everything is well defined)

↳ That is, if there are not "singularities", we can substitute (x,y) for (x_0,y_0).

• Examples:

a) $$\lim_{(x,y)\to(1,0)} \frac{x^2+2y+3}{x+y} = \frac{1^2+2\cdot 0+3}{1+0} = 4,$$

b) $$\lim_{(x,y)\to(1,2)} \frac{\sqrt{x^2+y^2}}{\ln(y+x)} = \frac{\sqrt{5}}{\ln 3},$$

c) $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \left(\frac{0}{0}\right)$$! It seems like an indeterminate form... but not really.

↳ $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)\cancel{(x-y)}}{\cancel{x-y}} = \lim_{(x,y)\to(0,0)} (x+y) = 0.$$

Remark: Notice that the function $f(x,y) = \frac{x^2-y^2}{x-y}$ is not defined at $(0,0)$, however the limit exists.

d) $$\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} \lim_{(x,y)\to(0,0)} e^y = 1.$$

$$^{*)} \lim_{x\to 0} \frac{\sin x}{x} = \left(\frac{0}{0}\right) \underset{\text{L'Hôpital}}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

c) $\lim\limits_{(x,y)\to(2,-4)} \dfrac{4+y}{x^2y - xy + 4x^2 - 4x} = \left(\dfrac{0}{0}\right) = \lim\limits_{(x,y)\to(2,-4)} \dfrac{4+y}{(x^2-x)y + 4(x^2-x)} =$

$= \lim\limits_{(x,y)\to(2,-4)} \dfrac{\cancel{4+y}}{(x^2-x)(\cancel{y+4})} = \lim\limits_{(x,y)\to(2,-4)} \dfrac{1}{x^2-x} = \dfrac{1}{2}.$

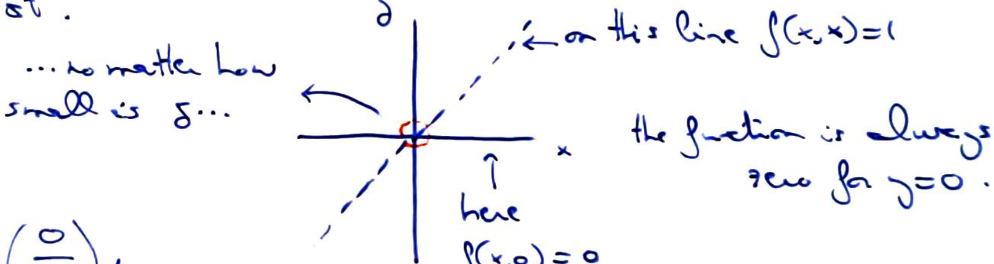
→ So, first thing to try is by direct substitution or simplifying a bit.

2) Show that the limit does not exist

The definition of limit in 1d required that the lateral limits were the same for the limit to exists.
Now, the limit in all directions must be the same

↳ <u>Conclusion</u>: If we find two different directions along which we obtain different limits, then the limit does not exist.

· <u>Examples</u>:



$\lim\limits_{(x,y)\to(0,0)} \dfrac{y}{x} = \left(\dfrac{0}{0}\right),$

Let's approach $(0,0)$ from the line $y=0$: $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=0}} \dfrac{y}{x} = 0$

Let's approach $(0,0)$ from the line $y=x$: $\lim\limits_{\substack{(x,y)\to(0,0)\\ y=x}} \dfrac{y}{x} = 1$

$\Rightarrow \nexists \lim ...$

• <u>Exercise</u>: Find the limit or show that it does not exist.

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = \left(\dfrac{0}{0}\right)$. so we try to show it doesn't exist.

Let's consider the limit through different lines $y = mx$, $m \in \mathbb{R}$

$$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{2xy}{x^2+y^2} = \lim\limits_{x\to 0} \frac{2mx^2}{x^2+m^2x^2} = \frac{2m}{1+m^2}.$$

So, in particular, for $m=0$ we would have $L=0$ but for $m=1$ we would obtain $L=1$. Thus, the limit does not exist.

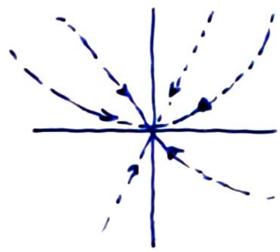
b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2} = \left(\dfrac{0}{0}\right)$

Let's try to see the limit through different lines: $y = mx$.

$$\lim\limits_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{2x^2y}{x^4+y^2} = \lim\limits_{x\to 0} \frac{2mx^3}{x^4+m^2x^2} = \lim\limits_{x\to 0} \frac{2mx}{x^2+m^2} = 0.$$

→ <u>Warning</u>! This <u>does not</u> imply that the limit is $L=0$.

↳ The limit has to be the same along <u>all directions</u>, which includes lines, paraboles, ...



↳ So this is $\boxed{\text{never}}$ an approach to find a limit (we simply can't consider all directions). It is a method to show that it does not exist, if we are lucky enough two find different directions giving different results.

We are going to still try to show that this particular limit doesn't exist.

Consider the limits along the parabolas $y = mx^2$:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx^2}} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2mx^4}{x^4+m^2x^4} = \frac{2m}{1+m^2} \rightarrow$$ The limit doesn't exist. (different values for different directions)

→ Hint: In general, it is useful to look at the denominator and make all the terms of the same "order" (same power):

$$\frac{2xy}{\underbrace{x^2+y^2}_{y\approx x}} \rightsquigarrow \text{"}y = mx\text{"}, \quad \frac{2x^2y}{\underbrace{x^4+y^2}_{y^2\approx x^4 \to y\approx x^2}} \rightsquigarrow \text{"}y = mx^2\text{"}$$

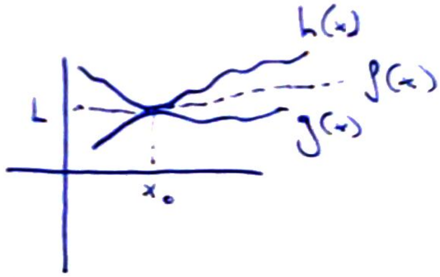
3) Show that the limit exists using the "Sandwich rule"

Theorem: Sandwich' Rule

If $\left.\begin{array}{l} g(x,y) \le f(x,y) \le h(x,y) \quad \forall (x,y) \neq (x_0,y_0) \text{ in a disk centered at } (x_0,y_0) \\ \lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = L \end{array}\right\}$

then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

→ Same as idea than in 1d:



• <u>Exercise</u>: Find the limit or show that it does not exist.

a) $\lim_{(x,y)\to(0,0)} y\sin\left(\frac{1}{x}\right)$

[<u>Remark</u>: Here, we cannot do
$\lim_{(x,y)\to(0,0)} y\sin\left(\frac{1}{x}\right) = \lim_{y\to 0} y \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$,
since the second one doesn't exist.]

We notice that $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ no matter what x is. Thus,

$$\left.\begin{array}{l} -y \leq y\sin\left(\frac{1}{x}\right) \leq y \quad (y \geq 0) \\ y \leq y\sin\left(\frac{1}{x}\right) \leq -y \quad (y \leq 0) \end{array}\right\} \text{ and } \lim_{(x,y)\to(0,0)} y = \lim_{(x,y)\to(0,0)} (-y) = 0,$$

thus $\lim_{(x,y)\to(0,0)} y\sin\left(\frac{1}{x}\right) = 0$.

b) $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \left(\frac{0}{0}\right)$.

(or $4x \leq \frac{4xy^2}{x^2+y^2} \leq 0$ if $x \leq 0$)

We notice that $0 \leq \frac{y^2}{x^2+y^2} \leq 1$, so $0 \leq 4x\frac{y^2}{x^2+y^2} \leq 4x$, therefore

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \lim_{x\to 0} 4x = 0$$

c) $\lim_{(x,y)\to(0,0)} \frac{\arctan(xy)}{xy}$, if we know that $1-\frac{x^2y^2}{3} < \frac{\arctan(xy)}{xy} < 1$.

Since $\lim_{(x,y)\to(0,0)} 1-\frac{x^2y^2}{3} = 1 = \lim_{(x,y)\to(0,0)} 1$, we can conclude that

$$\lim_{(x,y)\to(0,0)} \frac{\arctan(xy)}{xy} = 1.$$