$\boxed{\text{MATH 115 LECTURE 4}}$: Partial derivatives. (~Section 14.3)

Continuation of Lect. 3: Sect. 14.2

- "Methods" for computing limits
  - ① Directly by substitution.
  - ② Show that it doesn't exist (different values along different directions).

    Examples: Pages -28-, -29-

    $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{y^2+x^4}$ It is not sufficient to study all straight lines!
  - ③ Find the limit using Sandwich's Rule (pages -30-, -31-).
  - ④ "Funny ideas": Polar coordinates (new).

4) "Funny ideas": Change to polar coordinates.

If 1), 2), 3) don't work, you might try to compute the limit by changing to polar coordinates,

Example: $\lim_{(x,y)\to(0,0)} \frac{3x^4-y^2x}{x^2+y^2}$

As much as you try 2), you will always obtain the value 0, so you cannot prove that the limit doesn't exist.

Method 3) cannot be applied either. However, we can write

$$\lim_{(x,y)\to(0,0)} \frac{3x^4 - y^2 x}{x^2+y^2} = \lim_{\substack{r\to 0\\ \theta\in[0,2\pi]}} \frac{3r^4\cos^4\theta - r^3\sin^2\theta\cos\theta}{r^2} =$$

$$= \lim_{\substack{r\to 0\\ \theta\in[0,2\pi]}} \left(3r^2\cos^4\theta\right) - \lim_{\substack{r\to 0\\ \theta\in[0,2\pi]}} r\sin^2\theta\cos\theta = 0 - 0 = 0.$$

$$0 \le 3r^2\cos^4\theta \le 3r^2$$

$$-r \le r\sin^2\theta\cos\theta \le r$$

We end this section with the definition of continuity:

Def: A function $f(x,y)$ is continuous at (x_0,y_0) if

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)}$$

We say f is continuous in a region R if it is continuous at every point of R.

Remarks: There are three (hidden) points in the definition. For a function $f(x,y)$ to be continuous at (x_0,y_0), it is needed that:

1) f is defined at (x_0,y_0),

2) the limit of f at (x_0,y_0) exists,

3) $f(x_0,y_0)$ and $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ are the same.

• Examples: Where are these functions continuous?

a) $f(x,y) = \ln(x^2+y^2)$,

b) $f(x,y) = \dfrac{x+y}{x-y}$

c) $f(x,y) = \begin{cases} \dfrac{4xy^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0). \end{cases}$

Sol: Since we had that limits satisfy all the "standard rules" as in 1-d, we only need to consider the points where the functions are not defined or have jumps.

a) It is not defined at $\{(x,y)\in\mathbb{R}^2 : x^2+y^2=0\}$, so it cannot be continuous on that circle.
It is continuous everywhere else

b) Continuous at $\{(x,y)\in\mathbb{R}^2 : x\neq y\}$

c) For $(x,y)\neq(0,0)$ we have a standard function (fraction of polynomials), so we need to see if it is continuous or not at $(0,0)$.
In order to do so, we have to see if the limit at $(0,0)$ exists, and if it does, we have to check if it is equal to $f(0,0)=0$.

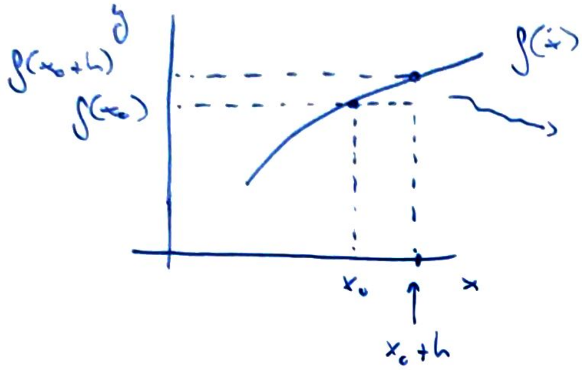
$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = \lim_{\substack{r\to0\\ \theta\in[0,2\pi]}} \frac{4r^3\cos\theta\sin^2\theta}{r^2} = \lim_{\substack{r\to0\\ \theta\in[0,2\pi]}} 4r\underbrace{\cos\theta\sin^2\theta}_{-1\le\ldots\le1} = 0.$$

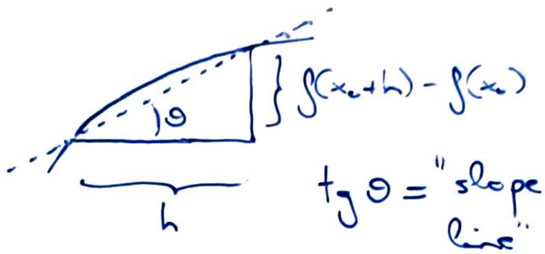
also by Sandwich' Rule, since $0 \le 4x\dfrac{y^2}{x^2+y^2} \le 4x$.

14.3| Partial derivatives.

Remember the definition of $\frac{df}{dx}$ for a function $f(x)$ and its meaning:

$$\frac{df}{dx}(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$





$tg\,\theta =$ "slope of dashed line" $= \frac{f(x_0+h)-f(x_0)}{h}$

That is, $f'(x)$ at x_0 is the "rate of change" of $f(x)$ when x is changing very closely to x_0.

- Partial derivatives

For a function in several variables, we can define in analogous way the "partial derivative" with respect to each variable:

Let's see in two variables: $f(x,y)$

→ Partial derivative with respect to x : $\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$

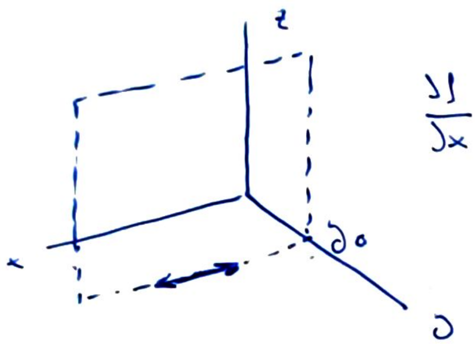
Remark: We may write $\frac{\partial f}{\partial x}$, f_x, $\partial_x f$, but never $\cancel{f'(x,y)}$.

What does this mean?

Notice that the variable y is <u>kept constant</u> while we change x.

So, if we think of the graph $z = f(x,y)$ (see pictures in slides of Lect. 4, page -7-, or in the book),

$\frac{\partial f}{\partial x}(x_0, y_0)$ is the slope of the tangent line to the curve we obtain when $z = f(x,y)$ intersects the plane $y = y_0$.



$\frac{\partial f}{\partial x}$ only changes f along the x-axis.

- Question (home): What is the definition of $\frac{\partial f}{\partial y}(x_0, y_0)$?

  And of $\frac{\partial f}{\partial z}(x_0, y_0, z_0)$ for $f(x,y,z)$?

- <u>Computation of partial derivatives</u>

In practice, to compute $\frac{\partial f}{\partial x}$, we look at $f(x,y)$ as a function of only the variable x, with y as if it was a fixed parameter.

Example: $f(x,y) = x^2 y + \sin(xy)$.

1) Find $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$.

2) Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ (at any point (x,y)).

Solution:

1) We can use the definition:

$$\frac{\partial f}{\partial x}(0,1) = \lim_{h \to 0} \frac{f(0+h,1) - f(0,1)}{h} = \lim_{h\to 0} \frac{(h^2 + \sin(h)) - 0}{h} = 1.$$

$$\frac{\partial f}{\partial y}(0,1) = \lim_{h \to 0} \frac{f(0,1+h) - f(0,1)}{h} = \lim_{h\to 0} \frac{0\cdot(1+h) + \sin(0\cdot(1+h)) - (0\cdot 1 + \sin(0\cdot 1))}{h} = 0.$$

2) To find $\frac{\partial f}{\partial x}(x,y)$, we have to think as if y is fixed.
It might help at first to write a new function only of x with y fixed:

$$g(x) = f(x,y_0) = x^2 y_0 + \sin(x y_0)$$

$$\hookrightarrow \frac{\partial f}{\partial x}(x,y_0) = g'(x) = 2x y_0 + \cos(x y_0) y_0 \to \frac{\partial f}{\partial x}(x,y) = 2xy + \cos(xy) y.$$

- We usually do this automatically in our head.

$$\frac{\partial f}{\partial y}(x,y) = x^2 + \cos(xy) x.$$

[Of course, if we substitute now $(x,y) = (0,1)$ we obtain the results in 1)].

• **Exercise:** Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

1) $f(x,y) = 2x^2 + 4xy$. 2) $f(x,y) = \sin(xy + y)$, 3) $f(x,y) = (1 + x^3)y^2$

4) $f(x,y) = e^x \sin(y)$

• **Exercise:** $\frac{\partial}{\partial y}\left(x^2 \tan(y) + y^2 + x\right)$ is : ?

$\frac{\partial}{\partial x}\left(e^y \sqrt{y^2 - 2\sin y}\right)$ is : ?

• **Exercise** || Implicit differentiation || (Recall calculus in 1-d)

Find $\frac{\partial z}{\partial x}$ if the equation $yz - \ln z = x + y$ defines a function of two variables $z = z(x,y)$.

Sol: