Part I: Core problems

These problems are from the book *Thomas' Calculus Early Transcendentals Custom Edition for the University of Pennsylvania.*

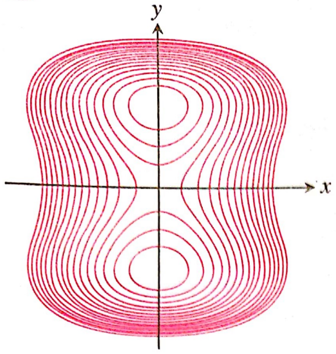
1. Section 14.1: Functions of several variables

1. Find the value of the function $f(x, y, z) = \dfrac{x - y}{y^2 + z^2}$ at
   - $(3, -1, 2)$,
   - $(1, \frac{1}{2}, -\frac{1}{4})$,
   - $(0, -\frac{1}{3}, 0)$,
   - $(2, 2, 100)$
2. Find and sketch the domain of $f(x, y) = \cos^{-1}(y - x^2)$.
3. Find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c (that is, we are sketching the contour map of f):

$$f(x, y) = x^2 + y^2, c = 0, 1, 4, 9, 16, 25.$$

4. For $f(x, y) = \sqrt{y - x}$,
   - find the domain,
   - find the range,
   - describe the level curves,
   - find the boundary of the domain,
   - determine if the domain is an open region, a closed region, or neither,
   - decide if the domain is bounded or unbounded.
5. Match each set of level curves with the appropriate function:

31.



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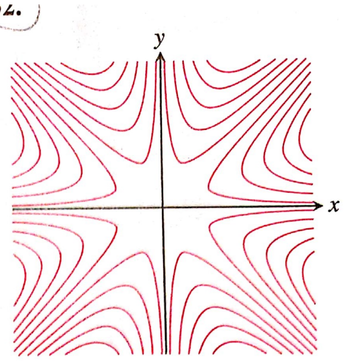

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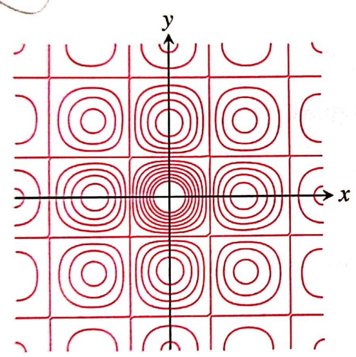

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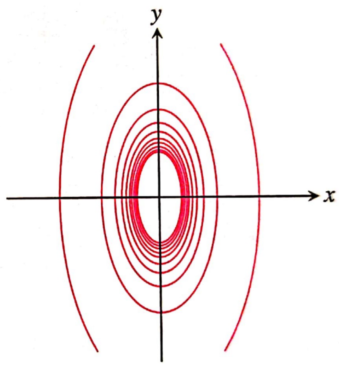

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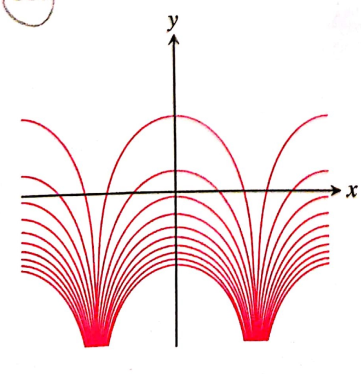

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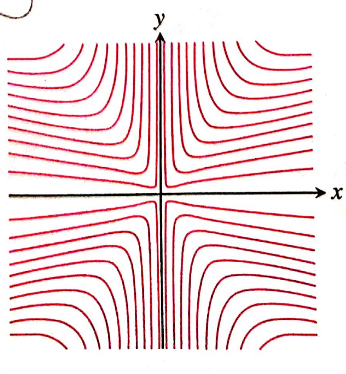
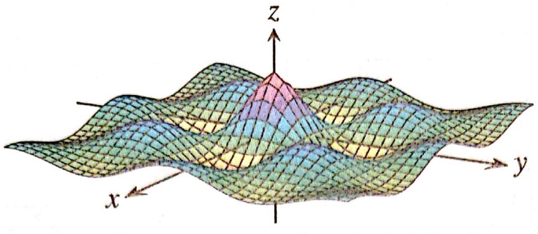
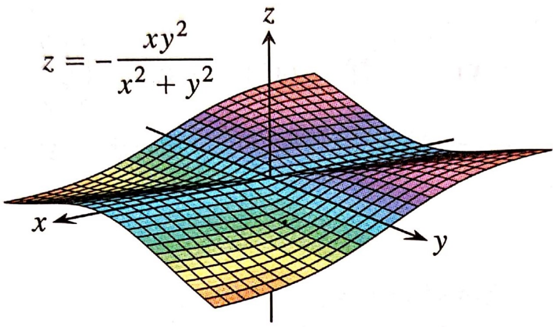


a.

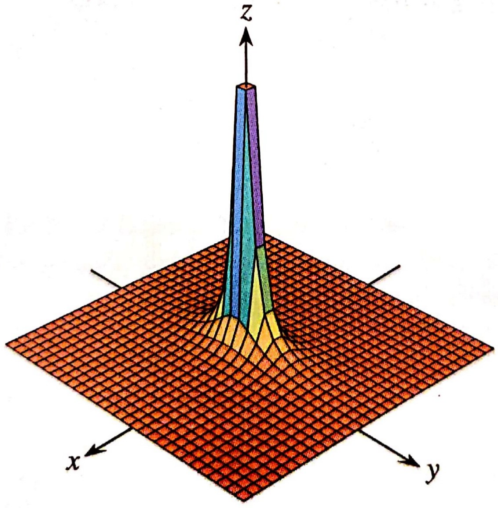


$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2 + y^2}/4}$

b.

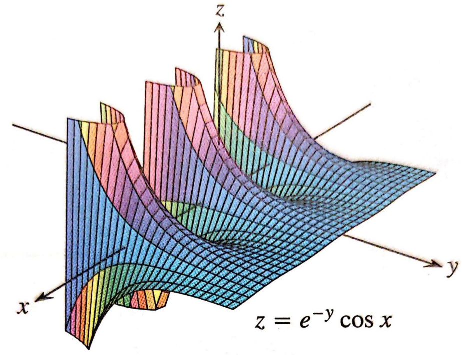


c.

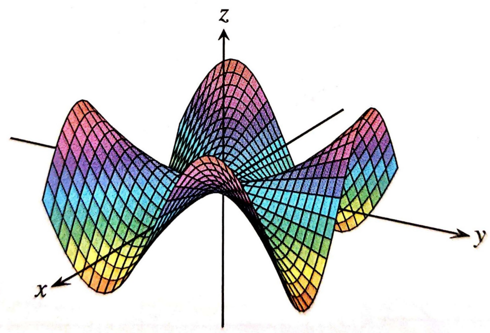


$z = \dfrac{1}{4x^2 + y^2}$

d.

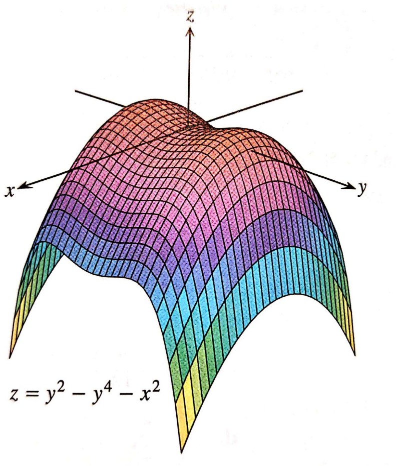


e.



$z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$

f.



6. Sketch the surface $z = f(x, y)$ and draw several level curves in the function's domain for $f(x, y) = x^2 + y^2$.

7. Find an equation for and sketch the graph of the level curve of the function $f(x, y) = \sqrt{x^2 - 1}$ that passes through the point $(1, 0)$.

8. Sketch a typical level surface for the function $f(x, y, z) = x + z$.

9. Find an equation for the level surface of the function $f(x, y, z) = \ln(x^2 + y + z^2)$ through the point $(-1, 2, 1)$.

10. Find and sketch the domain of $f(x, y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n$. Then find an equation for the level curve of the function passing through the point $(1, 2)$.

2. Section 14.2: Limits and Continuity in higher dimensions

1. Find the limits:
   - $\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$,
   - $\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$,
   - $\lim_{\substack{(x,y)\to(2,-4) \\ y\neq -4, x\neq x^2}} \frac{y + 4}{x^2 y - xy + 4x^2 - 4x}$,
   - $\lim_{P\to(\pi,\pi,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$,

2. At what points (x, y) in the plane is the function $f(x, y) = \frac{x + y}{x - y}$ continuous?

3. At what points (x, y) in the plane is the function $f(x, y) = \ln(x^2 + y^2)$ continuous?

4. By considering different paths of approach, show that the following functions have no limit as $(x, y) \to (0, 0)$:
   - $f(x, y) = -\frac{x}{\sqrt{x^2 + y^2}}$,
   - $f(x, y) = \frac{xy}{|xy|}$.

5. Show that the limit does not exists: $\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$.

6. Does knowing that

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4\cos\sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|}.$$

Give reasons for your answer.

7. Find the limit or show that the limit does not exist:

$$\lim_{(x,y)\to(0,0))} \frac{x^3-xy^2}{x^2+y^2}.$$

3. Section 14.3: Partial derivatives

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ for:

   - $f(x,y) = (xy-1)^2$,
   - $f(x,y) = \frac{1}{x+y}$,
   - $f(x,y) = x^y$,
   - $f(x,y) = x^2\,\ln(x)$,
   - $f(x,y) = \sum_{n=0}^{\infty}(xy)^n$,
   - $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$.

2. Find the partial derivative of the function with respect to each variable:

$$W(P,V,\delta,v,g) = PV + \frac{V\delta v^2}{2g}.$$

3. Find all the second-order partial derivatives of $f(x,y) = \arctan\left(\frac{y}{x}\right)$.

4. Verify that $w_{xy} = w_{yx}$ for $w(x,y) = x\sin y + y\sin x + xy$.

5. Let $w = f(x, y, z)$. Write the formal definition of the partial derivative with respect to z, $\partial f/\partial z$, at (x_0, y_0, z_0). *Use this definition* to find $\partial f/\partial z$ at $(1, 2, 3)$ for $f(x, y, z) = x^2yz^2$.

6. Find the value of $\partial z/\partial x$ at the point $(1, 1, 1)$ if the equation

$$xy + z^3x - 2yz = 0$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

7. Show that $f(x, y, z) = x^2 + y^2 - 2z^2$ satisfies the *Laplace equation*, that is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

8. Show that the function $w(x, t) = \sin(x + ct) + \cos(2x + 2ct)$ is a solution of the *wave equation*

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}.$$

9. Show that $u(x, t) = \sin(\alpha x)e^{-\beta t}$ satisfies the *heat equation*:

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$