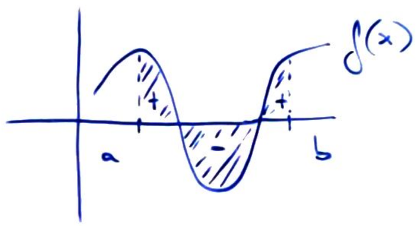
MATH 115 LECTURE 12 : Double and iterated integrals

(~Sec. 15.1, 15.2)

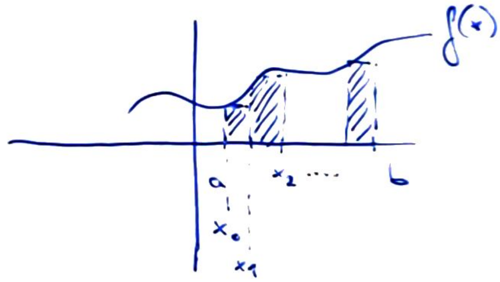
In 1-d, we know that the integral of a function represents the (signed) area below the curve:

$$\int_a^b f(x)\,dx$$



Formally, it was defined through Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$

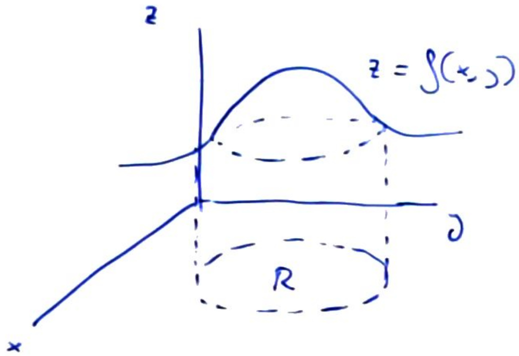


And the way we typically computed these integrals was using the Fundamental Theorem of Calculus: if we know a primitive of $f(x)$, that is, some $F(x)$ such that $F'(x) = f(x)$, then,

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a).$$

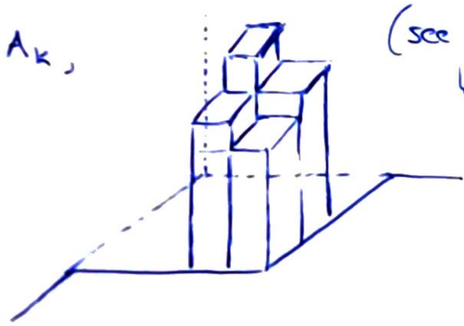
Analogously, integration in two variables gives the (signed) volume below the surface $z = f(x,y)$:

$$\iint_R f(x,y)\, dA$$



Formally, it can also defined by Riemann Sums:

$$\iint_R f(x,y)\, dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k ,$$

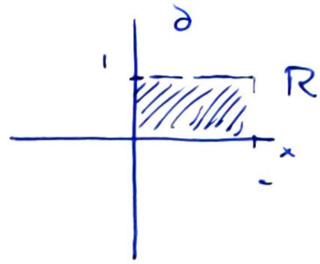


(see slides/book).

In practice, to compute these double integrals, we first define the region R in the form of inequalities and then we apply the Fundamental Theorem of Calculus twice.

Let's see some examples:

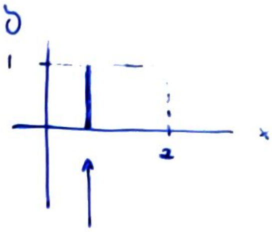
1) $\displaystyle\iint_R x^2 y \, dA$ where



Sol: The region R is defined by $0 \le x \le 2$,
$0 \le y \le 1$.

So we write

$$\iint_R x^2 y \, dA = \int_0^2 \left(\int_0^1 x^2 y \, dy \right) dx$$



for each $0 \le x \le 2$, we integrate in y from 0 to 1.
Then, we "add" these vertical lines to fill the rectangle.

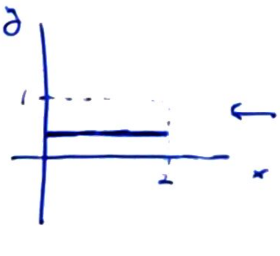
Now, we perform the inner integral with x as a parameter:

$$\int_0^1 x^2 y \, dy = x^2 \int_0^1 y \, dy = x^2 \left. \frac{y^2}{2} \right]_0^1 = x^2 \left(\frac{1}{2} - 0 \right) = \frac{x^2}{2},$$

so we obtain

$$\iint_R x^2 y \, dA = \int_0^2 \frac{x^2}{2} \, dx = \frac{1}{2} \left. \frac{x^3}{3} \right]_0^2 = \frac{8}{2 \cdot 3} = \frac{4}{3}.$$

Remark: In the same spirit, it is clear that we should obtain the same if we "add" horizontal lines:



← fix $0 \le y \le 1$, integrate from $x = 0$ to $x = 2$.

$$\iint_R x^2 y \, dA = \int_0^1 \left(\int_0^2 x^2 y \, dx \right) dy = \int_0^1 y \left(\int_0^2 x^2 \, dx \right) dy = \int_0^1 y \left. \frac{x^3}{3} \right]_0^2 dy =$$

$$= \int_0^1 y \, \frac{8}{3} \, dy = \frac{8}{3} \left. \frac{y^2}{2} \right]_0^1 = \frac{8}{3} \frac{1}{2} = \frac{4}{3}.$$

This is the so-called Fubini's Theorem:

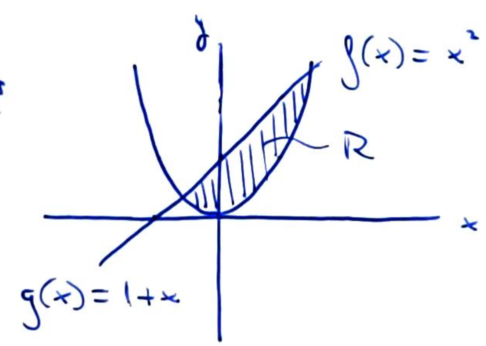
<u>Theorem</u>: Fubini's theorem [rectangular region]

If $f(x,y)$ is continuous in the region $a \leq x \leq b$, $c \leq y \leq d$, then,

$$\iint_R f(x,y) \, dA = \int_c^d \int_a^b f(x,y) \, dx \, dy = \int_a^b \int_c^d f(x,y) \, dy \, dx.$$

2) <u>Non-rectangular regions</u>

$$\iint_R x y \, dA \quad \text{where}$$



To perform the integral, we first need to describe the region R using inequalities for x and y.

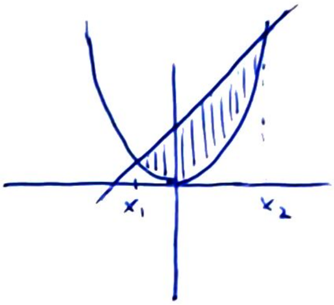
Let's study the points where $f(x) = x^2$ and $g(x) = 1 + x$ intersect:

$$f(x) = x^2 = 1 + x = g(x) \Leftrightarrow x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{1+4}}{2}$$

So the two points are

$$\begin{cases} x_1 = \dfrac{1-\sqrt{5}}{2} \to y_1 = \dfrac{3-\sqrt{5}}{2} \\ x_2 = \dfrac{1+\sqrt{5}}{2} \to y_2 = \dfrac{3+\sqrt{5}}{2} \end{cases}$$

So R can be described by



$$\frac{1-\sqrt{5}}{2} \le x \le \frac{1+\sqrt{5}}{2}, \quad x^2 \le y \le 1+x$$

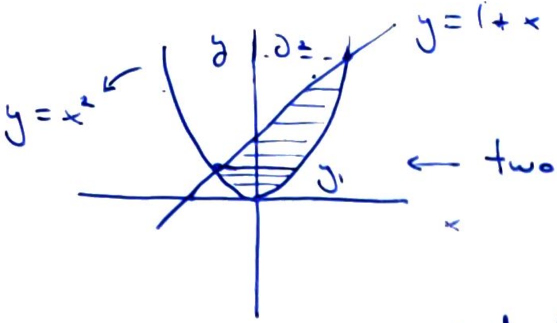
Therefore,

$$\iint_R xy \, dA = \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} \left(\int_{x^2}^{1+x} xy \, dy \right) dx = \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} x \left(\int_{x^2}^{1+x} y \, dy \right) dx =$$

$$= \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} x \left. \frac{y^2}{2} \right]_{x^2}^{1+x} dx = \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} x \left(\frac{(1+x)^2}{2} - \frac{x^4}{2} \right) dx = \dots = \frac{5\sqrt{5}}{8}.$$

<u>Question</u>: How can we write this double integral integrating in x first?

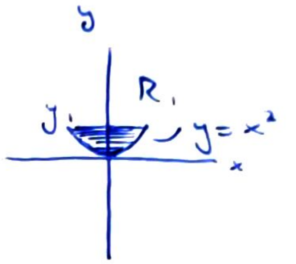
Recall that $\iint_R xy \, dA$ represents the volume under the surface $z = xy$ on the region R, so both integrals should be equal.

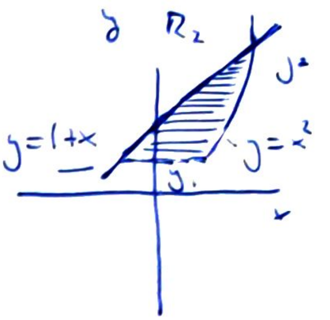


← two regions: for $0 \le y \le y_1$, the variable x varies from $-\sqrt{y}$ to $\sqrt{y}$.

and for $y_1 \le y \le y_2$, x goes from $y-1$ to $\sqrt{y}$.

↳ $$\iint_R xy\,dA = \iint_{R_1} xy\,dA + \iint_{R_2} xy\,dA,$$





• $$\iint_{R_1} xy\,dA = \int_0^{y_1}\left(\int_{-\sqrt{y}}^{\sqrt{y}} xy\,dx\right)dy =$$

$$= \int_0^{\frac{3-\sqrt{5}}{2}} y\left(\int_{-\sqrt{y}}^{\sqrt{y}} x\,dx\right)dy = \int_0^{\frac{3-\sqrt{5}}{2}} y\,\left.\frac{x^2}{2}\right]_{-\sqrt{y}}^{\sqrt{y}} dy = \int_0^{\frac{3-\sqrt{5}}{2}} y\cdot 0\,dy = 0.$$

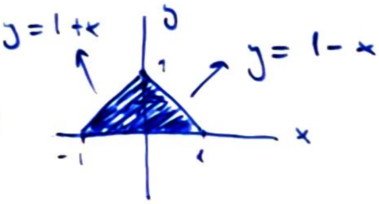
• $$\iint_{R_2} xy\,dA = \int_{y_1}^{y_2}\left(\int_{y-1}^{\sqrt{y}} xy\,dx\right)dy =$$

$$= \int_{\frac{3-\sqrt{5}}{2}}^{\frac{3+\sqrt{5}}{2}} y\,\left.\frac{x^2}{2}\right]_{y-1}^{\sqrt{y}} dy = \int_{\frac{3-\sqrt{5}}{2}}^{\frac{3+\sqrt{5}}{2}} y\left(\frac{y}{2} - \frac{(y-1)^2}{2}\right)dy = \frac{5\sqrt{5}}{8}.$$

→ [See general Fubini's theorem in book]

• <u>Exercices:</u>

1) Calculate $\iint_R y \, dA$ where R is



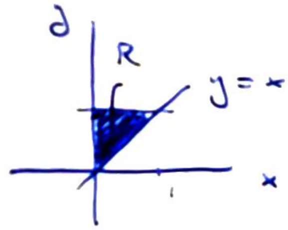
<u>Sol:</u> It's easier here to integrate in x first ("horizontally")

$$\iint_R y \, dA = \int_0^1 \left(\int_{y-1}^{1-y} y \, dx \right) dy = \int_0^1 y \left(\int_{y-1}^{1-y} dx \right) dy =$$

$$= \int_0^1 y \left(1-y-(y-1) \right) dy = \int_0^1 y \left(1-y-y+1 \right) dy = \frac{1}{3}.$$

[<u>Question:</u> What if we wanted to integrate in y first?]

2) Calculate $\iint_R e^{y^2} \, dA$ where



<u>Sol:</u>

Here we don't really have a choice! If we try integrating in y first, we get stuck:

$$\iint_R e^{y^2} dA = \int_0^1 \left(\int_x^1 e^{y^2} dy \right) dx \quad ??$$

However, we can check that

$$\iint_R e^{y^2}\,dA = \int_0^1 \left(\int_0^y e^{y^2}\,dx \right) dy = \int_0^1 e^{y^2} \left(\int_0^y dx \right) dy =$$

$$= \int_0^1 e^{y^2}\, y\, dy = \frac{1}{2} e^{y^2} \Big]_0^1 = \frac{1}{2}(e-1) .$$