$\boxed{\begin{array}{l}\text{MATH 115}\\ \text{LECTURE 13}\end{array}}$: Sets and counting techniques (~Chapters 1, 2)

1) Sets

• Def: A <u>set</u> is a collection of objects, called elements.

For example,

$$A = \{a, b, c\}, \quad B = \{1, 3, 6\},$$

$$C = \{A, B\} = \{\{a, b, c\}, \{1, 3, 6\}\},$$

$$D = \{\text{numbers with three digits}\},$$

$$E = \{(x, y) : 0 \le y \le x^2\}.$$

• Def: A set A is a <u>subset</u> of a set B if every element of A also belongs to B.

In this case, we also say that A is contained in B, and we write $A \subseteq B$.

<u>Example</u>: $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{\{1, 2\}, 3\}$

Here, $A \subseteq B$ $\left[1 \in A \to 1 \in B;\ 2 \in A \to 2 \in B;\ 3 \in A \to 3 \in B \right]$.

Also, $A \subseteq A$ (this is always true).

But $C \not\subseteq B$, since $\{1, 2\}$ $\overbrace{\text{is not an element of } B}^{\text{in this case the elements of}}$. B are numbers.

[the set $\{1, 2\}$ is a subset of B though].

- Two special sets:

1) Empty set, $\emptyset$ → set with no objects.

↳ For any set A, $\emptyset \subseteq A$.

2) Universal set, $\mathcal{U}$: this depends on the problem and is usually understood from context.

↳ For example, $\{a,b\}$, $\{a,b,d\}$, $\{c,f\}$ are sets of letters. Here, the universal set would be the alphabet.

- <u>Operations with sets</u>

<u>Def</u>: The union of two sets A and B is the set of all elements that belong to A or B, $A \cup B$.

<u>Def</u>: The intersection of two sets A and B is the set of all elements which belong to both A and B, $A \cap B$.

<u>Example</u>: $A = \{1,2,3,4,5\}$, $B = \{1,3,5,7,9\}$

↳ $A \cup B = \{1,2,3,4,5,7,9\}$

$A \cap B = \{1,3,5\}$

Def: The complement of a set A is the set of all elements in U (universal set) which do not belong to A, A^c.

Def: The difference between a set A and a set B is the set of elements that belong to A but not to B, $A \setminus B$.

Example: $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$

Find A^c, B^c, $A \cup B$, $(A \cup B)^c$, $A \cap B$.

Sol: $A^c = \{4, 5\}$, $B^c = \{2, 4\}$

$A \cup B = \{1, 2, 3, 5\}$, $(A \cup B)^c = \{4\}$

$A \cap B = \{1, 3\}$

- We summarize here the previous definitions in mathematical notation:

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A^c = \{x : x \in U, x \notin A\}$

$A \setminus B = \{x : x \in A, x \notin B\}$

- <u>Counting elements</u>

We denote $n(A)$ the number of elements in the set A.

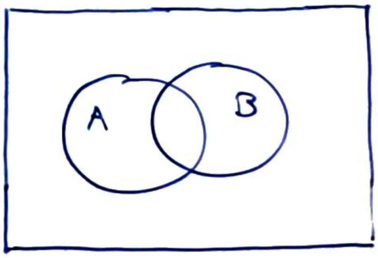
We want to find relations between $n(A)$, $n(B)$, $n(A \cap B)$, ...

For this, <u>Venn diagrams</u> are very useful.

↳ Consider two sets A, B and the universal set U.

We represent them as follows:

1)



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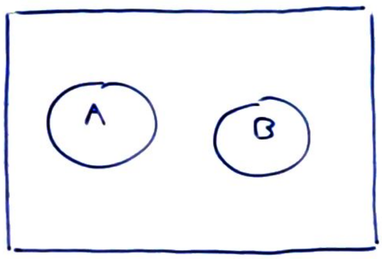

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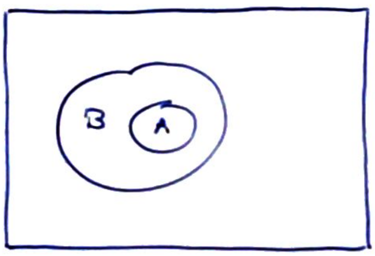


In case 1), there are some common elements between A and B.

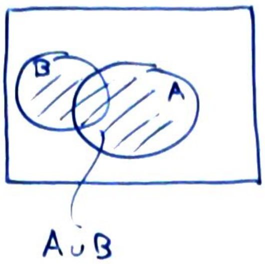
2), the sets A and B are <u>disjoint</u> (no common elements, that is, $A \cap B = \emptyset$).

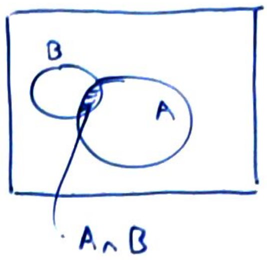
3) A is contained in B, $A \subseteq B$.

→ Inclusion-Exclusion principle:

$$\boxed{n(A\cup B) = n(A) + n(B) - n(A\cap B)}$$

easy to visualize:

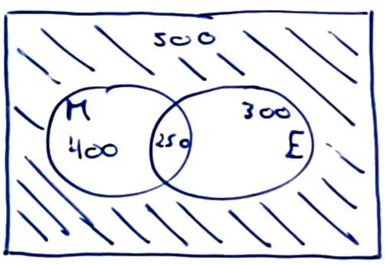




• <u>Exercise</u>: Of the 500 students in some college, 400 are taking a math course, 300 are taking an economics course, and 250 are taking both a math course and an economics one.

How many are taking neither a math course nor an econ one?

<u>Sol</u>:



We see that we want to find $n((M\cup E)^c)$

It is always true that $n(A)+n(A^c)=n(U)$, since A and A^c are disjoint by definition (thus, $A\cap A^c=\emptyset$ and so $n(A\cap A^c)=0$).

We have that $n((M\cup E)^c) = n(\Omega) - n(M \cup E)$.

Now,

$$n(\Omega) = 500,$$

$$n(M\cup E) = n(M) + n(E) - n(M\cap E) = 400 + 300 - 250 = 450,$$

therefore,

$$n((M\cup E)^c) = 500 - 450 = 50.$$

2) Counting techniques

• <u>Multiplication principle</u>

<u>Example</u>: Suppose a restaurant has 3 different appetizers and 2 different entrees. How many ways can you order an appetizer and an entree?

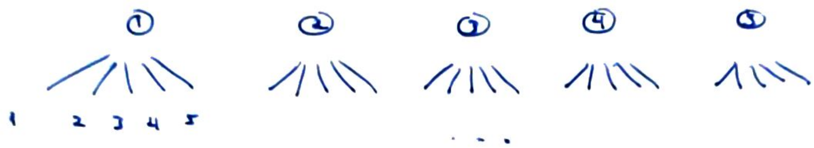
<u>Sol</u>: First the appetizer → A_1 $\quad$ A_2 $\quad$ A_3

Once chose, two options → E_1 E_2 $\quad$ E_1 E_2 $\quad$ E_1 E_2.

So in total 6 different ways: $6 = 3\cdot 2$.

<u>Example</u>: A gambler plans to fly from Philadelphia to Las Vegas on Monday, lose his money on Tuesday, and fly back to Philadelphia on Wednesday. The are 4 flights from Philadelphia to Las Vegas on Monday and 5 for the return. How many choices of flight bookings

<u>Sol</u>: We first choose the flight on Monday and for each case we have 5 choices for the return.



So in total, 25 choices: $25 = 5 \cdot 5$.

<u>Multiplication principle</u>: Suppose an event E can occur in m ways and, independently of this event, an event F can occur in n ways. Then, the combination of events E and F can occur in $m \cdot n$ ways.

↳ analogously for more events.

Exercise: Using the multiplication principle, find how many ways you can order the letters a, b, c, d, e, f.

Sol:
We have six positions to fill with those letters,

___ ___ ___ ___ ___ ___

To fill the first one, we have 6 choices. After that, we only have 5 letters left, so to fill the second position there are only 5 choices.
We continue this process to find that we have

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ ways.

• Def: The factorial of a natural number n is

$$n! = n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1.$$

For convenience, we also define $0! = 1$.

Exercise: We flip a coin 5 times and write the results down. How many possible outcomes are there?

Sol: Notice that each event (fliping a coin) is independent of the others. So we have 5 events, each one with two possible outcomes. Therefore, the total possible outcomes are

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32.$$

- Permutations

Def: Consider n objects. A permutation of the n objects taking r at a time is any arrangement of any $r \leq n$ of these objects in a given order.

The number of permutations of n objects taking r at a time is denoted by $P(n, r)$.

Example: Objects: a, b, c, d (four letters)

↳ $abcd$, $bacd$, $abdc$ are permutations

↳ abc, acb, bac are permutations taking three at a time.

• How can we know $P(n,r)$?

<u>Example</u>: Find the number of three-letter words that can be form with five letters a, b, c, d, e.

<u>Sol</u>: This corresponds to $P(5,3)$.

We can calculate it using the multiplication principle.

We have three positions to be filled:

$$\underset{\substack{\uparrow \\ 5 \text{ options}}}{\_\_}\ \_\_\ \_\_$$

After the first letter, the second one can be chosen in 4 ways, and for the last one there are 3 ways.

Therefore, $P(5,3) = 5 \cdot 4 \cdot 3 = 60$.

We can now generalize this. If we have n different objects and take r at a time, to compute the number of ways we can do this in order we proceed as before:

$$\underbrace{\_\_\ \_\_\ \_\_\ \_\_\ \cdots\ \_\_}_{r \text{ positions}}$$

1^{st} position → n choices.

2^{nd} position → $n-1$ choices.

3^{rd} position → $n-2$ choices

$\vdots$

At the last position (the r^{th} one), there are $n-(r-1)$ choices.

In total,

$$P(n,r) = n(n-1)(n-2)\dots(n-(r-1)).$$

We can write this using factorials:

$$P(n,r) = n(n-1)(n-2)\dots(n-r+1)\cdot\frac{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1} =$$

$$= \frac{n!}{(n-r)!}$$

- In summary, $\boxed{P(n,r) = \frac{n!}{(n-r)!}}$

<u>Exercise</u>: Consider a club with 5 members. How many ways can they choose a president and a vice president?

<u>Sol</u>:

$$P(5,2) = \frac{5!}{3!} = \frac{5\cdot 4\cdot \cancel{3!}}{\cancel{3!}} = 20.$$

president $\downarrow$ vice president $\swarrow$

__ __

$\uparrow$ 5 choices $\uparrow$ 4 choices

• Remark: Notice that $P(n,n)$ is the number of ways we can order n objects, $P(n,n) = n!$.

• Permutations with repeated objects

Example: How many five-letter words can be form with the letters from the word BABBY?

Sol: Let's first label the B's: $B_1 A B_2 B_3 Y$. Then, we can form $P(5,5) = 5! = 120$ words, like the following:

→ $\underline{B_1}\,\underline{B_2}\,\underline{B_3}AY$, $\underline{B_1}\,\underline{B_3}\,\underline{B_2}AY$, $\underline{B_2}\,\underline{B_1}\,\underline{B_3}AY$, $\underline{B_2}\,\underline{B_3}\,\underline{B_1}AY$, $\underline{B_3}\,\underline{B_1}\,\underline{B_2}AY$, $\underline{B_3}\,\underline{B_2}\,\underline{B_1}AY$

→ $\underline{B_1}A\underline{B_2}\,\underline{B_3}Y$, $\underline{B_1}A\underline{B_3}\,\underline{B_2}Y$, $\underline{B_2}A\underline{B_1}\,\underline{B_3}Y$, $\underline{B_2}A\underline{B_3}\,\underline{B_1}Y$, $\underline{B_3}A\underline{B_1}\,\underline{B_2}Y$, $\underline{B_3}A\underline{B_2}\,\underline{B_1}Y$

⋮

But each row corresponds to the same word, so what we really want to know is the number of rows.

Since there are 120 "words" in total, and each row has $6 = 3!$, the number of rows is: $120 = 6 \cdot$ n° of rows ⇒

⇒ The answer is $\frac{120}{6} = 20$.