MATH 115
LECTURE 5

The Chain Rule. (~ Sec. 14.4)

Continuation of Lecture 4:

1. Exercise: Level curves and partial derivatives.
2. Exercise: Partial derivatives and tangents. (slides)
3. Exercise: First approach to implicit differentiation (page -37-, Lecture 4).

Some remarks/theorem that relates partial derivatives and continuity:

• Remark 1: A function can have partial derivatives at a point without the function being continuous (impossible in 1d!).

↳ Example: $f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$

that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at $(x,y)=(0,0)$, but f is discontinuous there.

Indeed:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0,$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0,$$

but

$\lim_{(x,y)\to(0,0)} f(x,y)$ $\underbrace{\text{does not exist}}$, so $f(x,y)$ not continuous at $(0,0)$.

↳ limit along $x=0$ is 1
limit along $x=y$ is 0.

• <u>Remark 2</u>: If the partial derivatives of $f(x,y)$ exist and are continuous in an open region Ω, then f is continuous

[Theorem] in Ω

(more is true, the function is "differentiable"; we will talk about that later.)

• <u>Remark 3</u>: If a function $f(x,y)$ and its partial derivatives

[Theorem] f_x, f_y, f_{xy}, f_{yx} all exist and are continuous in an open region Ω, then

$$\| \; f_{xy}(x,y) = f_{yx}(x,y) \; \| \quad \forall (x,y) \in \Omega .$$

→ In this course, almost always the functions will be "smooth" (i.e., all derivatives exist and are continuous).

↳ So the "mixed derivative theorem" is useful to compute second derivatives faster.

14.4 The Chain Rule

In 1-d, if we know the derivative of $f(x)$ and $g(x)$, we know the derivative of $f\circ g$ (composition of f and g):

$$(f\circ g)(x) = f(g(x)) \to \frac{d}{dx}(f\circ g)(x) = f'(g(x))\,g'(x).$$

We can write this in a more convenient way for several variables:

$$w = f(x) \text{ with } x = g(t) \to \frac{dw}{dt} = \frac{df}{dx}\frac{dg}{dt}$$

Basically, we are saying that w is a function of t, and x is just an intermediate variable: $w(t) = f(g(t))$.

Theorem: Chain Rule with one independent variable ("curves")

Let $f(x,y)$ and $x = x(t)$, $y = y(t)$ be "differentiable" functions. Then,

$w(t) = f(x(t), y(t))$ is differentiable in t and

$$w'(t) = \frac{dw}{dt}(t) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt}(t).$$

↳ We can also write:

$$w'(t) = f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t))\,y'(t),$$

or $$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ (careful with this one: where is each derivative evaluated?).

Remark: Same thing for more intermediate variables

$w = f(x,y,z)$ with $x = x(t)$, $y = y(t)$, $z = z(t)$, then

$$\frac{dw}{dt} \equiv \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

just notation

(properly, $w'(t) = f_x(x(t), y(t), z(t))\, x'(t) +$
$+ f_y(x(t), y(t), z(t))\, y'(t) + f_z(x(t), y(t), z(t))\, z'(t))$.

Exercises: Let $w(t) = f(x(t), y(t))$ or $w(t) = f(x(t), y(t), z(t))$. Find $w'(t)$:

1) $f(x,y) = x^2 + y^2$, $x(t) = \cos t$, $y(t) = \sin(t)$

2) $f(x,y,z) = \frac{x}{z} + \frac{y}{z}$, $x(t) = \cos^2 t$, $y(t) = \sin^2 t$, $z(t) = \frac{1}{t}$.

Sol:

1) $w'(t) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$,

$f_x(x,y) = 2x$, $f_y(x,y) = 2y$, $x'(t) = -\sin(t)$, $y'(t) = \cos(t)$, thus

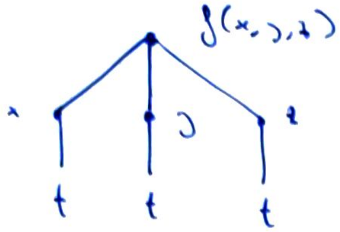
$w'(t) = 2\cos(t)(-\sin(t)) + 2\sin(t)\cos(t) = 0$.

Can you "visualize" the result in geometric terms?

Remark: We can also do this working in 1-d!

$w(t) = f(x(t), y(t)) = \cos^2(t) + \sin^2(t) = 1 \rightsquigarrow w'(t) = 0$.

It may help to think in the "tree diagram" to remember the chain rule:



$$\frac{df}{dt} =$$

2) $f_x(x,y,z) = \frac{1}{z}$, $f_y(x,y,z) = \frac{1}{z}$, $f_z = \frac{-1}{z^2}(x+y)$

$x'(t) = 2\cos(t)(-\sin(t))$, $y'(t) = 2\sin(t)\cos(t)$, $z'(t) = \frac{-1}{t^2}$,

so

$$w'(t) = t\left(-2\cancel{\cos(t)\sin(t)}\right) + t\cdot 2\cancel{\sin(t)\cos(t)} - t^2\left(\cos^2 t + \sin^2 t\right)\frac{-1}{t^2} = 1.$$

Theorem: Chain Rule with two independent variables ("surfaces")

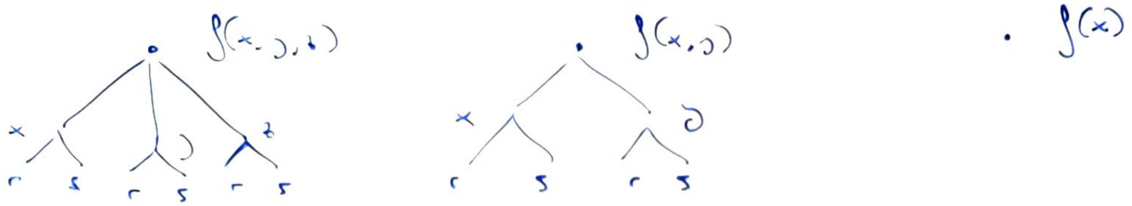
Let $f(x,y,z)$ and $x = x(r,s)$, $y = y(r,s)$, $z = z(r,s)$ be differentiable. Then

$w(r,s) = f(x(r,s), y(r,s), z(r,s))$ is differentiable and

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}.$$

- Analogously if there are only one or two intermediate variables.



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$\cdots$

- <u>Exercise</u>:

1) Find $\dfrac{\partial f}{\partial u}$ if $f(x,y) = x^2 + \dfrac{y}{x}$, $x = u - 2v + 1$, $y = 2u + v - 2$.

2) Find $\dfrac{\partial w}{\partial t}, \dfrac{\partial w}{\partial s}$ if $w = f(s^3 + t^2)$ and $f'(x) = e^x$.

3) <u>Polar coordinates</u>: Let $w = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$.

   a) Show that $\dfrac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta$,

   $$\frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta.$$

   b) Solve equations in a) to express f_x and f_y in terms of $\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial \theta}$.

4) $T(x,y,z) = e^{-(x^2+y^2+z^2)}$ gives the temperature at any point in a room. The movement of a particle is given by

$$\left.\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin t \\ z(t) &= t \end{aligned}\right\}$$

What is the rate of change of the temperature for the particle at the point $(0, 1, \frac{\pi}{2})$?

• <u>Implicit differentiation</u>: If $F(x,y)=0$ defines y as a function of x, $y=f(x)$, then $\frac{dy}{dx} = -\frac{F_x}{F_y}$ (for $F_y \neq 0$).

<u>Also</u>: If $F(x,y,z)=0$ defines $z=f(x,y)$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

<u>Remark</u>: We don't need this formulas, but they can be useful for checking the results.

↳ Recall a previous exercise: $yz - \ln z = xy$, $\frac{\partial z}{\partial x}$?