MATH 115
LECTURE 6

: ~~Directional derivatives and gradient vector.~~

Chain Rule: exercises. (~~Sec 14.5~~)

Continuation of Lect. 5: Chain Rule

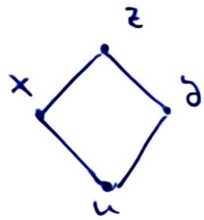
Exercises

1] Let $z = \sin(xy) + x\sin(y)$, $x = u^2+v^2$, $y = uv$.
Find $\dfrac{\partial z}{\partial u}$.

Sol: $z(u,v) = \sin\big(x(u,v)\,y(u,v)\big) + x(u,v)\sin\big(y(u,v)\big)$, so

$$\frac{\partial z}{\partial u}(u,v) = \frac{\partial z}{\partial x}\big(x(u,v),y(u,v)\big)\frac{\partial x}{\partial u}(u,v) + \frac{\partial z}{\partial y}\big(x(u,v),y(u,v)\big)\frac{\partial y}{\partial u}(u,v)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$



We need to compute z_x, z_y, x_u, y_u:

$$\begin{cases} z_x(x,y) = y\cos(xy) + \sin(y), \quad z_y(x,y) = x\cos(xy) + x\cos(y), \\ x_u(u,v) = 2u, \quad y_u(u,v) = v \end{cases}$$

Finally,

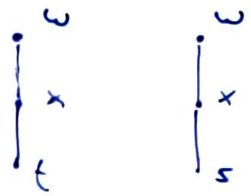
$$\frac{\partial z}{\partial u}(u,v) = \Big[uv\cos\big(uv(u^2+v^2)\big) + \sin(uv)\Big]2u + \Big[(u^2+v^2)\cos\big(uv(u^2+v^2)\big) + (u^2+v^2)\cos(uv)\Big]v.$$

2] Let $w = f(s^3+t^2)$ and $f'(x) = e^x$. Find $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$.

<u>Sol</u>: By the chain rule,

$$\frac{\partial w}{\partial t}(s,t) = \frac{\partial f}{\partial x}(x(s,t))\frac{\partial x}{\partial t}(s,t),$$

$$\frac{\partial w}{\partial s}(s,t) = \frac{\partial f}{\partial x}(x(s,t))\frac{\partial x}{\partial s}(s,t).$$



Now,

f only depends on one variable, so $\frac{\partial f}{\partial x} \equiv \frac{df}{dx} \equiv f'$.

The intermediate variable here is $x = s^3+t^2$, thus

$$\frac{\partial x}{\partial t}(s,t) = 2t, \quad \frac{\partial x}{\partial s}(s,t) = 3s^2, \quad \frac{\partial f}{\partial x}(x(s,t)) = f'(x(s,t)) = e^{x(s,t)} = e^{s^3+t^2},$$

so finally $\quad \frac{\partial w}{\partial t}(s,t) = e^{s^3+t^2}\,2t, \quad \frac{\partial w}{\partial s}(s,t) = e^{s^3+t^2}\,3s^2.$

3] Polar coordinates.

Consider $w = f(x,y)$ and the change of variables $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$.

3.1) Show that $\begin{cases} w_r = f_x\cos\theta + f_y\sin\theta \\ \frac{1}{r}w_\theta = -f_x\sin\theta + f_y\cos\theta \end{cases}$

3.2) Find f_x, f_y in terms of f_r, f_θ.

3.3) Show that $(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$.

Sol.:

3.1) Apply the chain rule to $w(r,\theta) = f(x(r,\theta), y(r,\theta))$:

$$w_r(r,\theta) = f_x(x(r,\theta), y(r,\theta))\frac{\partial x}{\partial r}(r,\theta) + f_y(x(r,\theta), y(r,\theta))\frac{\partial y}{\partial r}(r,\theta) =$$

$$= f_x \cos\theta + f_y \sin\theta, \checkmark$$

$$w_\theta(r,\theta) = f_x(x(r,\theta), y(r,\theta))\frac{\partial x}{\partial \theta}(r,\theta) + f_y(x(r,\theta), y(r,\theta))\frac{\partial y}{\partial \theta}(r,\theta) =$$

$$= f_x\cdot(-r\sin\theta) + f_y\cdot r\cos\theta \Rightarrow$$

$$\Rightarrow \frac{1}{r}w_\theta = -f_x \sin\theta + f_y\cos\theta. \checkmark$$

3.2) Let's solve this system for f_x and f_y:

$$\left.\begin{aligned} w_r &= f_x\cos\theta + f_y\sin\theta \\ \frac{1}{r}w_\theta &= -f_x\sin\theta + f_y\cos\theta \end{aligned}\right\} \xrightarrow{(1^{st}\text{ equation}) + \frac{\cos\theta}{\sin\theta}(2^{nd}\text{ equation})}$$

$$\Rightarrow w_r - \frac{\cos\theta}{\sin\theta}\frac{1}{r}w_\theta = \cancel{f_x\cos\theta} - \cancel{\frac{\cos\theta}{\sin\theta}\sin\theta f_x} + f_y\sin\theta + \frac{\cos\theta}{\sin\theta}\cos\theta f_y \Leftrightarrow$$

$$\Leftrightarrow w_r - \frac{\cos\theta}{\sin\theta}\frac{1}{r}w_\theta = f_y\left(\sin\theta + \frac{\cos^2\theta}{\sin\theta}\right) = f_y\frac{1}{\sin\theta} \Leftrightarrow$$

$$\Leftrightarrow \left\| f_y = w_r\sin\theta - \frac{\cos\theta}{r}w_\theta \right\|.$$

Going back, we find that

$$w_r = f_x\cos\theta + \left(w_r\sin\theta - \frac{\cos\theta}{r}w_\theta\right)\sin\theta \Rightarrow$$

$$\Rightarrow f_x\cos\theta = w_r - w_r\sin^2\theta + \frac{\cos\theta\sin\theta}{r}w_\theta = w_r\cos^2\theta + \frac{\cos\theta\sin\theta}{r}w_\theta \Leftrightarrow$$

$$\Leftrightarrow \boxed{f_x = w_r\cos\theta + \frac{\sin\theta}{r}w_\theta}$$

3.3) We can do this using part 3.2) (substitute f_x, f_y in terms of w_r, w_θ and do computations), but it is easier if we notice that

$$\left.\begin{aligned} w_r &= f_x\cos\theta + f_y\sin\theta \\ \frac{1}{r}w_\theta &= -f_x\sin\theta + f_y\cos\theta \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (w_r)^2 + \frac{1}{r^2}(w_\theta)^2 = (f_x\cos\theta + f_y\sin\theta)^2 + (-f_x\sin\theta + f_y\cos\theta)^2 =$$

$$= (f_x)^2\cos^2\theta + (f_y)^2\sin^2\theta + \cancel{2f_xf_y\cos\theta\sin\theta} +$$

$$+ (f_x)^2\sin^2\theta + (f_y)^2\cos^2\theta \cancel{- 2f_xf_y\cos\theta\sin\theta} =$$

$$= (f_x)^2(\cos^2\theta + \sin^2\theta) + (f_y)^2(\cos^2\theta + \sin^2\theta) = (f_x)^2 + (f_y)^2. \checkmark$$

4] Let $T(x,y,z) = e^{-(x^2+y^2+z^2)}$ be the temperature at a point (x,y,z) in a room. A particle moves along the helix

$$\left.\begin{array}{l} x(t) = \cos(t), \\ y(t) = \sin(t), \\ z(t) = t \end{array}\right\}.$$

Find the rate of change of the temperature of the particle when it passes through $(0, 1, \frac{\pi}{2})$.

<u>Sol</u>:

At $(x(t), y(t), z(t)) = (0, 1, \frac{\pi}{2})$ the value of the parameter t is $t = \frac{\pi}{2}$ (since $z(t) = t = \frac{\pi}{2}$).

Therefore, what we need to compute is

$$\left.\frac{dT}{dt}(x(t), y(t), z(t))\right|_{t=\frac{\pi}{2}}.$$

Using the chain rule,

$$\frac{dT}{dt}(x(t), y(t), z(t)) = \frac{\partial T}{\partial x}(x(t), y(t), z(t))\, x'(t) + \frac{\partial T}{\partial y}(x(t), y(t), z(t))\, y'(t) + \frac{\partial T}{\partial z}(x(t), y(t), z(t))\, z'(t).$$

So at $t = \frac{\pi}{2}$,

$$\left.\frac{dT}{dt}(x(t), y(t), z(t))\right|_{t=\frac{\pi}{2}} = \underset{x(\frac{\pi}{2})=0}{0} + \underset{y'(\frac{\pi}{2})=0}{0} - 2\,\frac{\pi}{2}\, e^{-(1+\frac{\pi^2}{4})} \cdot 1 = -\pi\, e^{-\frac{4+\pi^2}{4}}.$$

5] Implicit differentiation (see page -44-).

Assume $x e^{y} + y e^{z} + 2\ln(x) - 2 - 3\ln(2) = 0$ define z as a function of x and y. Then find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(1, \ln(2), \ln(3))$.

Sol:

1st way

- Write $z = z(x,y)$ in the equation: $x e^{y} + y e^{z(x,y)} + 2\ln(x) - 2 - 3\ln(2) = 0$
- Take $\frac{\partial}{\partial x}$ and use the chain rule:

$$e^{y} + y e^{z(x,y)} \frac{\partial z}{\partial x}(x,y) + \frac{2}{x} = 0$$

- Solve for $\frac{\partial z}{\partial x}$: $\frac{\partial z}{\partial x} = -\left(e^{y} + \frac{2}{x}\right)\frac{1}{y} e^{-z}$
- Plug in $(x,y,z) = (1, \ln(2), \ln(3))$: $\frac{\partial z}{\partial x}(1, \ln(2)) = -(2+2)\frac{1}{\ln(2)}\frac{1}{3} =$

$$= -\frac{4}{3\ln(2)}.$$

→ Analogously for $\frac{\partial z}{\partial y}$.

2nd way

Use the formula in page -44-:

Define $F(x,y,z) = xe^{y} + ye^{z} + 2\ln(x) - 2 - 3\ln(z)$. Then,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^{y} + \frac{2}{x}}{ye^{z}} = -\left(e^{y} + \frac{2}{x}\right)\frac{1}{y}e^{-z}$$ (of course, same result).

• <u>Remark</u>: Why this formula works?

Consider the equation $F(x,y,z)=0$, and assume this equation defines $z = z(x,y)$. Then,

$F(x, y, z(x,y)) = 0$.

So we can take derivative in x by applying the chain rule:

$$\frac{d}{dx}\Big(F(x, y, z(x,y))\Big) = \frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \;//.$$

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