MATH 115
LECTURE 7

: Directional derivatives and gradient vector.

(~ Sec. 14.5).

14.5 | Directional derivatives and gradient vector.

- **Def:** The derivative of $f(x,y)$ at $P_0(x_0,y_0)$ in the direction of (the unit vector) $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is the number

$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+hu_1, y_0+hu_2) - f(x_0,y_0)}{h}.$$

Example: Find $D_{\vec{u}} f$ for $f(x,y) = x^2+y^2$, $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$ at $P_0(1,2)$.

$$D_{\vec{u}} f(1,2) = \lim_{h\to 0} \frac{f\left(1+h\frac{1}{\sqrt{2}}, 2+h\frac{1}{\sqrt{2}}\right) - f(1,2)}{h} =$$

$$= \lim_{h\to 0} \frac{(1+h/\sqrt{2})^2 + (2+h/\sqrt{2})^2 - (1^2+2^2)}{h} = \lim_{h\to 0} \frac{1+\frac{h^2}{2}+2\frac{h}{\sqrt{2}}+4+\frac{h^2}{2}+4\frac{h}{\sqrt{2}}-5}{h} =$$

$$= \frac{6}{\sqrt{2}} = 3\sqrt{2}.$$

- <u>Interpretation and computation of $D_{\vec{u}}f$</u>

Notice that if
$$\begin{cases} \vec{u} = \vec{i} \longrightarrow D_{\vec{u}}f(x,y) \equiv \dfrac{\partial f}{\partial x}(x,y). \\ \vec{u} = \vec{j} \longrightarrow D_{\vec{u}}f(x,y) \equiv \dfrac{\partial f}{\partial y}(x,y). \end{cases}$$

$D_{\vec{u}}f(x,y)$ gives the change of rate of f at (x,y) when we move along the direction given by $\vec{u}$ (see slides).

This allows us to compute $D_{\vec{u}}f$ using the chain rule: we study how $f(x,y)$ change along the line define by $\vec{u}$ that passes through the point:

$D_{\vec{u}}f(x_0,y_0)$? Line $\left.\begin{array}{l} x(t) = x_0 + tu_1 \\ y(t) = y_0 + tu_2 \end{array}\right\}$ $(\vec{u} = (u_1, u_2)$ <u>unit</u> vector$)$

$$\hookrightarrow D_{\vec{u}}f(x_0,y_0) = \frac{d}{dt} f(x(t), y(t))\Big|_{t=0} = \left(\frac{\partial f}{\partial x}(x(t),y(t))\,x'(t) + \frac{\partial f}{\partial y}(x(t),y(t))\,y'(t)\right)\Big|_{t=0} =$$

$$= f_x(x_0,y_0)\,x'(0) + f_y(x_0,y_0)\,y'(0) =$$

$$= f_x(x_0,y_0)\,u_1 + f_y(x_0,y_0)\,u_2 = (f_x(x_0,y_0), f_y(x_0,y_0))\cdot(u_1,u_2) \equiv$$

$$\equiv (f_x(x_0,y_0)\,\vec{i} + f_y(x_0,y_0)\,\vec{j})\cdot(u_1\vec{i} + u_2\vec{j}).$$

Notice that we have reduced the computation of any directional derivative to finding the partial derivatives $\|$.

• Def: Gradient of f

In 2-d, $\nabla f(x,y) = (f_x(x,y), f_y(x,y))$,

In 3-d, $\nabla f(x,y,z) = (f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$.

• Summary:

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}}$$ ($\vec{u}$ unit vector, $|\vec{u}|=1$).

• Example: Let $f(x,y) = xy$, $P_0(1,1)$, $\vec{u} = 2\vec{i} + \vec{j}$.

1) Find $\nabla f(1,1)$ and $D_{\vec{u}} f$ at P_0.

2) Plot the level curve of $f(x,y)$ that goes through P_0 together with $\nabla f(1,1)$.

Sol:

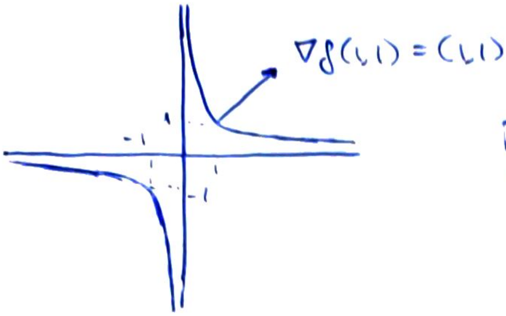
1) $\nabla f(x,y) = (f_x(x,y), f_y(x,y)) = (y, x) \rightarrow \nabla f(1,1) = (1,1)$.

Now, $\vec{u} = 2\vec{i} + \vec{j}$ is not a unit vector $\rightarrow \vec{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{\sqrt{5}}(2\vec{i} + \vec{j})$ is.

$D_{\vec{v}} f(1,1) = \nabla f(1,1) \cdot \vec{v} = (1,1) \cdot \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = \frac{3}{\sqrt{5}}$.

2) $f(x,y) = c = xy = f(1,1) = 1 \rightarrow c = 1 \rightarrow$

$\rightarrow$ Level curve through $(1,1)$ is $xy = 1$.



<u>Remark</u>: $\nabla f(1,1)$ is perpendicular to the level curve at $(1,1)$.

- <u>Properties</u>

$D_{\vec{u}} f = \nabla f \cdot \vec{u} \Rightarrow$ $D_{\vec{u}} f = |\nabla f| \underbrace{|\vec{u}|}_{=1} \cos\theta = |\nabla f| \cos\theta$, where θ is the angle between ∇f and $\vec{u}$.

Therefore,

1) f increases fastest in the direction given by ∇f: $\vec{u} = \frac{\nabla f}{|\nabla f|}$ (that is, $\cos\theta = 1$).

2) f decreases fastest when $\vec{u} = -\frac{\nabla f}{|\nabla f|}$ (i.e., $\cos\theta = -1$).

3) f doesn't change in directions perpendicular to ∇f ($\cos\theta = 0$).

• <u>Example</u>: Find the directions in which $f(x,y) = x^2 y + e^{xy}\sin(y)$ increases/decreases most rapidly at $P_0(1,0)$. Find the derivative at that point in those directions.

<u>Sol</u>: $$\nabla f(x,y) = \left(2xy + y e^{xy}\sin(y),\ x^2 + x e^{xy}\sin(y) + e^{xy}\cos(y)\right),$$

$$\nabla f(1,0) = (0,2)$$

Thus, $f(x,y)$ at $(1,0)$ | increases most rapidly along $\vec{u} = (0,1)$,
| decreases most rapidly along $\vec{u} = (0,-1)$,

and $$D_{(0,1)} f(x,y) = \nabla f(0,1) \cdot (0,1) = 2$$

$$D_{(0,1)} f(x,y) = \nabla f(0,1) \cdot (0,1) = -2$$

→ In general,

$D_{\vec{u}} f(x,y) = |\nabla f(x,y)|$ in direction of most rapidly increase

...

→ If we were asked direction along which $f(x,y)$ doesn't change at $(1,0)$, those would be $\vec{u} = (1,0)$ or $\vec{u} = (-1,0)$.

- Gradient and level curves

Consider the level curves of $f(x,y)$, $f(x,y)=c$.

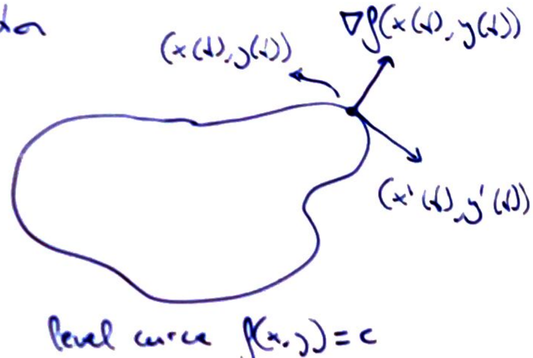
We can parametrize one of those curves by $(x(t), y(t))$.
It is clear that $f(x,y)$ is constant along this curves,

$f(x(t), y(t)) = c$, thus

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t) =$$

$$= \underbrace{\nabla f(x(t), y(t))}_{\text{Gradient of } f} \cdot \underbrace{(x'(t), y'(t))}_{\text{tangent vector}} = 0$$

That is, at any point on a level curve $f(x,y)=c$, the gradient of f is perpendicular to the level curve.



↳ Application: Find equations for the tangent and normal lines to a given curve.

[See Lecture 1].

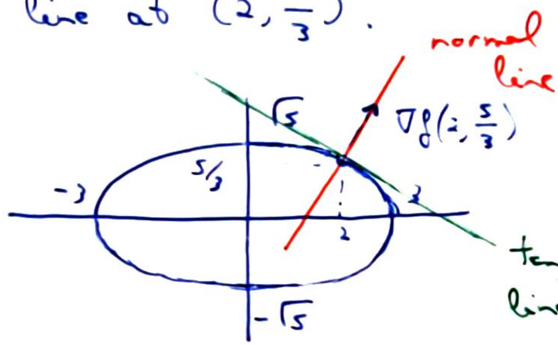
• <u>Exercise</u>: (Important)

Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$.

1) Find an equation for the tangent line at $(2, \frac{5}{3})$.

2) Find an equation for the normal line at $(2, \frac{5}{3})$.

<u>Sol</u>:



Define $f(x,y) = \frac{x^2}{9} + \frac{y^2}{5}$.

Then, the ellipse is the level curve $f(x,y) = 1$.

Therefore, $\nabla f(2, \frac{5}{3}) = \left(\frac{2}{9} \cdot 2, \frac{2}{5} \frac{5}{3}\right) = \left(\frac{4}{9}, \frac{2}{3}\right)$ is perpendicular to the ellipse at $(2, \frac{5}{3})$.

→ Tangent line at $(2, \frac{5}{3})$: Line through $(2, \frac{5}{3})$ perpendicular to $(\frac{4}{9}, \frac{2}{3})$

↳ $$\frac{4}{9}(x-2) + \frac{2}{3}\left(y - \frac{5}{3}\right) = 0$$

→ Normal line at $(2, \frac{5}{3})$: Line through $(2, \frac{5}{3})$ in the direction of $(\frac{4}{9}, \frac{2}{3})$

$$\begin{cases} x(t) = 2 + \frac{4}{9} t \\ y(t) = \frac{5}{3} + \frac{2}{3} t \end{cases}$$

• In general, for $f(x,y)=c$:

→ Tangent line at $P(x_0, y_0)$:

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0$$

→ Normal line at $P(x_0,y_0)$:

$$\left.\begin{aligned} x(t) &= x_0 + f_x(x_0,y_0)\,t \\ y(t) &= y_0 + f_y(x_0,y_0)\,t \end{aligned}\right\}$$