MATH 115
LECTURE 8 : Tangent planes and Linearization

(~ Sec. 14.6)

14.6 / Tangent planes and Differentials.

We ended last lecture by noticing that if $f(x,y)$ is a function of two variables and $f(x,y)=c$ a level curve, then we can find equations for the tangent line and normal line at a point (x_0, y_0) by using the gradient of f.

- Similarly, think of a level surface $f(x,y,z)=0$.

For any curve $(x(t), y(t), z(t))$ on the level surface it is clear that

$$\frac{d}{dt} f(x(t), y(t), z(t)) = 0$$, so using the chain rule

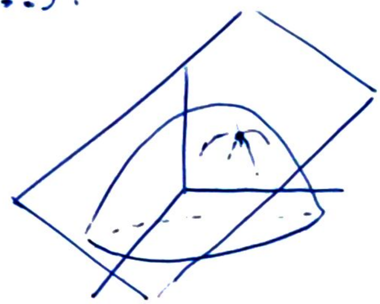
$$f_x(x(t), y(t), z(t))\, x'(t) + f_y(x(t), y(t), z(t))\, y'(t) + f_z(x(t), y(t), z(t))\, z'(t) = 0.$$

We can write this as $\nabla f(x(t), y(t), z(t)) \cdot \underbrace{(x'(t), y'(t), z'(t))}_{\text{tangent vector to the curve}} = 0$

That is, ∇f is a vector perpendicular to <u>all</u> curves on the level surface passing through the point.

• Def: The tangent plane at $P_0(x_0, y_0, z_0)$ to the surface $f(x,y,z) = c$ is the plane through P_0 and perpendicular to $\nabla f(x_0, y_0, z_0)$.

Remark: The plane is "tangent" to all the curves on the surface passing through P_0.



• Def: The normal line to the surface $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$ is the line through P_0 parallel to $\nabla f(x_0, y_0, z_0)$.

→ Using the definitions, it is clear that (see lecture 1):

Equation for tangent plane: $f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$.

Parametric equations for the normal line:
$$\left.\begin{aligned} x &= x_0 + f_x(x_0,y_0,z_0)\,t \\ y &= y_0 + f_y(x_0,y_0,z_0)\,t \\ z &= z_0 + f_z(x_0,y_0,z_0)\,t \end{aligned}\right\}$$

• **Example:** Find equations for the tangent plane and normal to the surface $x^2 + y^2 = 9 - z$ at $P_0(1,2,4)$.

Sol:

We define $f(x,y,z) = x^2 + y^2 + z$, and consider the level surface $f(x,y,z) = 9$.

Therefore,

→ Tangent plane: $f_x(1,2,4)(x-1) + f_y(1,2,4)(y-2) + f_z(1,2,4)(z-4) = 0$

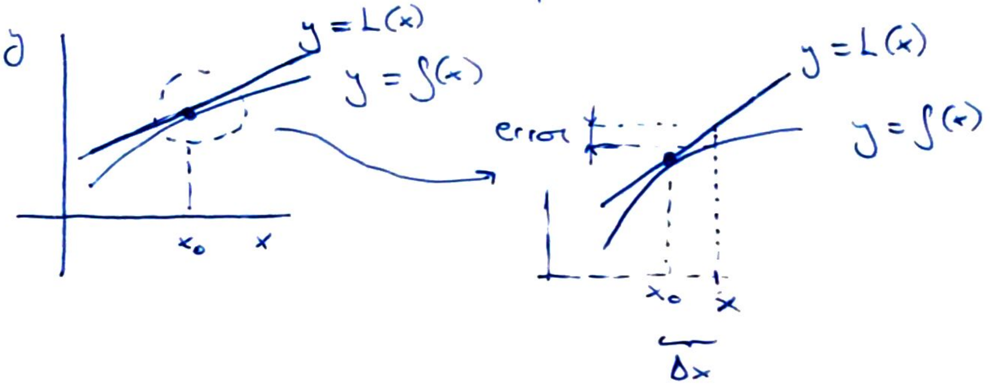
$$\hookrightarrow \; 2(x-1) + 4(y-2) + (z-4) = 0.$$

→ Normal line:

$$\left.\begin{aligned} x &= 1 + 2t, \\ y &= 2 + 4t, \\ z &= 4 + t. \end{aligned}\right\}$$

- <u>Approximation of a function, change and error</u>

Remember that in 1-d, the tangent line gives an approximation of the function $f(x)$ near a point x_0:



Recall that the equation of the tangent line is

$$y = L(x) = f(x_0) + f'(x_0)(x - x_0).$$

→ So near x_0, we can say that

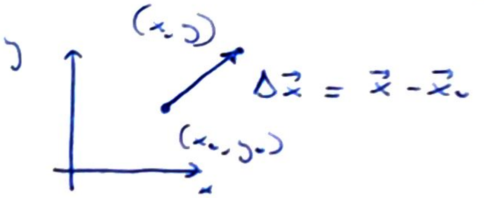
$$f(x) \underset{\text{approx.}}{\approx} L(x) = f(x_0) + f'(x_0)(x - x_0).$$

In other words, by knowing $f'(x_0)$, we can approximate how $f(x)$ changes near x_0:

$$f(x) - f(x_0) = \boxed{\Delta f \approx f'(x_0)\,\Delta x} = f'(x_0)(x - x_0)$$

• Analogously, consider a function $f(x,y)$ and a point $\vec{x}_0 = (x_0, y_0)$.

If we move to a nearby point $\vec{x} = (x,y)$, we are moving from $\vec{x}_0$ to $\vec{x}$ – that is, we are moving along the direction given by $\Delta\vec{x} = \vec{x} - \vec{x}_0$ a distance $|\Delta\vec{x}| = |\vec{x} - \vec{x}_0|$:



so $\vec{u} = \dfrac{\Delta\vec{x}}{|\Delta\vec{x}|}$ (unit vector).

That is, we have that

$$f(\vec{x}) - f(\vec{x}_0) = \boxed{\Delta f \approx D_{\vec{u}} f(\vec{x}_0)\, |\Delta\vec{x}|} = D_{\vec{u}} f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$$

$$\left(\vec{u} = \frac{\Delta\vec{x}}{|\Delta\vec{x}|} \right)$$

This is completely analogous to 1d!

• <u>Example</u>: Estimate how much the value of $f(x,y) = xy$ will change if we move a distance of 0.5 units from $P_0(1,1)$ along $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

<u>Sol</u>: $\Delta f \approx D_{\vec{u}} f(1,1)\, 0.5 = \nabla f(1,1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) 0.5 = \frac{1}{\sqrt{2}}$.

Notice that we can write the previous formula as follows:

$$\Delta f \approx D_{\vec{u}} f(\vec{x}_0)\,|\Delta\vec{x}| = \nabla f(\vec{x}_0)\cdot\vec{u}\,|\Delta\vec{x}| = \nabla f(\vec{x}_0)\cdot\frac{\Delta\vec{x}}{\cancel{|\Delta\vec{x}|}}\,\cancel{|\Delta\vec{x}|}$$

$$\vec{u} = \frac{\Delta\vec{x}}{|\Delta\vec{x}|}$$

Therefore, we have found that

$$f(x,y) - f(x_0,y_0) \equiv \boxed{\Delta f \approx \nabla f(x_0,y_0)\cdot\Delta\vec{x}} \equiv \nabla f(x_0,y_0)\cdot(x-x_0,\, y-y_0).$$

More explicitly, we are approximating $f(x,y)$ by

$$f(x,y) \approx L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

· <u>Def:</u> Given $f(x,y)$, we call the linearization of $f(x,y)$ at (x_0,y_0) to the function given by $L(x,y)$:

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

→ The linearization gives a good approximation of the function near (x_0,y_0).

Notice that it corresponds to the tangent plane at (x_0,y_0)!

→ Some applied problems (in class).