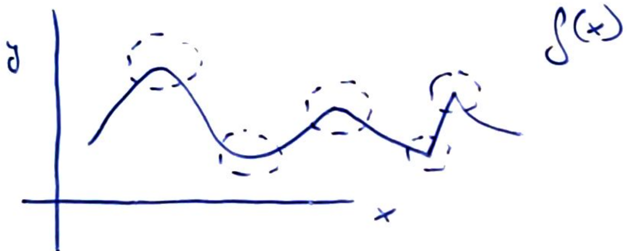
MATH 115
LECTURE 9 : Extreme Values and Saddles points.

Today we want to find the maximum and minimum values of a given function in several variables. As in 1-d, this will allow us to later solve optimization problems.

• <u>Extrema in the whole space</u>

We start with the simplest case: our domain has no boundaries. Then the extrema happen at "critical points". Think of 1-d:



↪ all those are "critical" points:

→ $f'(x) = 0$

or

→ $f'(x)$ does not exist at the point.

Notice that all these points are "local minima / maxima". Let's give the definition in 2-d: (analogous in 3-d)

• Def: Let $f(x,y)$ be defined in a region that contains the point (x_0, y_0). Then,

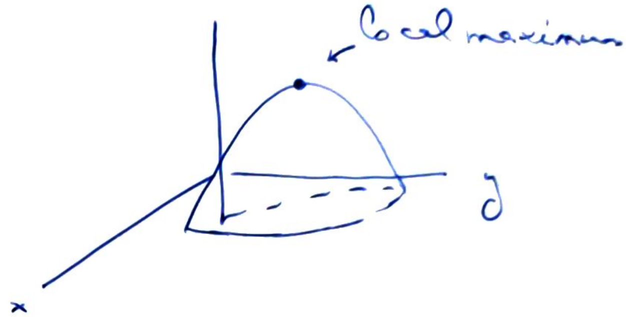
i) $f(x_0,y_0)$ is a local maximum if

$f(x_0,y_0) \geq f(x,y)$ for all (x,y) in an (arbitrarily small) open disk centered at (x_0,y_0).

ii) $f(x_0,y_0)$ is a local minimum if

$f(x_0,y_0) \leq f(x,y)$ for ...

(See slides)



In 1-d we had a necessary condition for a local extrema:

$f'(x) = 0$ or the derivative doesn't exist.

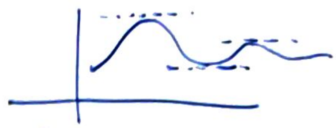
Analogously,

• Theorem: 1st derivative test

If $f(x,y)$ has a local extrema at an interior point (x_0,y_0), and if the partial derivatives exist, then

$$\boxed{\nabla f(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0)) = (0,0)}$$

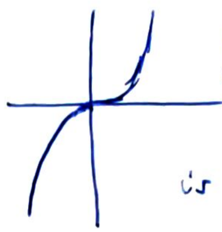
Interpretation: If f has a local extreme, the tangent plane has to be horizontal!



(see slides for 2-d)

Remark: This condition is not sufficient!

Recall in 1-d $f(x) = x^3$



$f'(0)=0$ but at $x=0$ there is no local min/max!

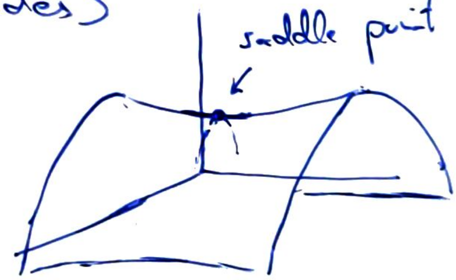
Same happens in 2-d:

- Def: Saddle point

Let (x_0, y_0) be a critical point of $f(x,y)$ [That is, $\nabla f(x_0,y_0)=(0,0)$ or it doesn't exist]

Then, $f(x,y)$ has a saddle point at (x_0,y_0) if in every disk centered at (x_0,y_0) there point for which $f(x,y) > f(x_0,y_0)$ and others for which $f(x,y) < f(x_0,y_0)$.

(see slides)



→ Question: How does level curve look like around local min/max and saddle points?

So, if $\nabla f(x,y) = (0,0)$, we need to study the "second derivative".

<u>Theorem</u>: 2^{nd} derivative test

Suppose that $\nabla f(x_0, y_0) = (0,0)$. Then,

1) If $\left.\begin{array}{l} f_{xx} < 0 \\ f_{xx} f_{yy} - (f_{xy})^2 > 0 \end{array}\right\}$ at (x_0, y_0), then (x_0, y_0) is a local max.

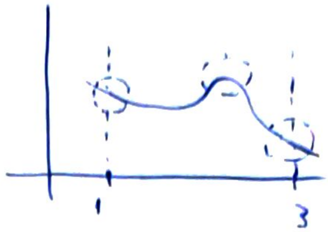
2) If $\left.\begin{array}{l} f_{xx} > 0 \\ f_{xx} f_{yy} - (f_{xy})^2 > 0 \end{array}\right\}$ at (x_0, y_0), then (x_0, y_0) is a local min.

3) If $f_{xx} f_{yy} - (f_{xy})^2 < 0$, then (x_0, y_0) is a saddle point.

4) If $\underbrace{f_{xx} f_{yy} - (f_{xy})^2}_{\text{called discriminant or Hessian of } f} = 0$, the test is inconclusive.

• Extrema in regions with boundaries

If there are boundaries, extrema can occur at critical points and at the boundaries:



So, procedure in general:

① Find and list all interior points where f might have local min/max (that is, the critical points).

↳ Look for solutions of $\left.\begin{array}{l} f_x(x,y)=0 \\ f_y(x,y)=0 \end{array}\right\}$

or points where the partial derivatives don't exist.

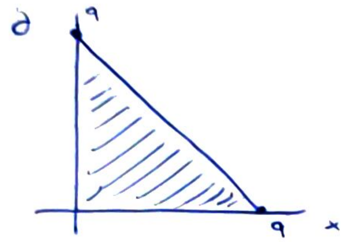
② Look at the boundary points and find candidates for max/min.

Finally, if we want to find the "absolute extrema", evaluate the function at all the candidates and choose the winners.

• Example: Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

on the region bounded by $x=0$, $y=0$, $y=9-x$.

Sol: 

① Critical points

Notice that $f(x,y)$ is differentiable everywhere, so only possibility is points where $\nabla f(x,y) = (0,0)$:

$$\left.\begin{aligned} f_x(x,y) &= 2-2x = 0 \\ f_y(x,y) &= 2-2y = 0 \end{aligned}\right\} \Leftrightarrow (x,y) = (1,1).$$

So, only one critical point, at $(1,1)$, with value $\boxed{f(1,1)=4}$

② Boundaries

Let's study each side of the triangle separately:

→ $\underline{y=0}$: Here, $f(x,y) = f(x,0) = 2+2x-x^2$.
$(0 \le x \le 9)$

So we have to find min./max. of $g(x) = 2+2x-x^2$ on $[0,9]$

↳ Interior points: $g'(x)=0 \Leftrightarrow x=1 \to f(1,0) = 3$.

↳ Boundaries: $g(0) = 2$, $g(9) = -61$.

So candidates $\boxed{f(1,0)=3,\ f(0,0)=2,\ f(9,0)=-61}$

→ $\underline{x=0}$: Repeat same argument to discover that the candidates are
$(0 \le y \le 9)$

$$\boxed{f(0,0)=2,\ f(0,9)=-61,\ f(0,1)=3}$$

→ $\underline{y=9-x}$: $f(x,9-x) = 2+2x+2(9-x)-x^2-(9-x)^2 =$
$(0 \le x \le 9)$

$$= -61+18x-2x^2.$$

So we need to study $g(x) = -61+18x-2x^2$ on $[0,9]$.

↳ $g'(x)=0 \Leftrightarrow x = 9/2 \rightsquigarrow \boxed{f\left(\frac{9}{2},\frac{9}{2}\right) = \frac{-41}{2}}$

Boundaries $\boxed{f(0,9)=-61,\ f(9,0)=-61}$

Finally, list of candidates: $f(1,1)=4$, $f(1,0)=3=f(0,1)$,
$f(0,0)=2$, $f(9,0)=-61=f(0,9)$
$f\left(\frac{9}{2},\frac{9}{2}\right) = \frac{-41}{2}$.

⟶ So,

the absolute minimum is -81, at $(0,9)$ and $(9,0)$.

the absolute maximum is 4, at $(1,1)$.

- Additional question: Determine if the critical points are minima, maxima or saddle points.

  ↳ Only need to study $(1,1)$.