$\boxed{\text{Part III}}$: Linear Algebra.

(Book: Linear Algebra, by Jim Hefferon). ↳ online.

$\boxed{\text{Chapter 1}}$: Linear Systems

Example: Toluene C_7H_8 and nitric acid HNO_3 mix to produce trinitrotoluene $C_7H_5O_6N_3$, and some water (H_2O) as a byproduct. In what proportion should we mix them?

Sol: Mass conservation tells us that the number of atoms of each element must remain the same, thus

$$x\, C_7H_8 + y\, HNO_3 \to z\, C_7H_5O_6N_3 + w\, H_2O$$

with

$$\left.\begin{array}{ll} C \rightsquigarrow & 7x = 7z \\ H \rightsquigarrow & 8x + y = 5z + 2w \\ N \rightsquigarrow & y = 3z \\ O \rightsquigarrow & 3y = 6z + w \end{array}\right\}$$ Linear system of 4 equations with 4 unknowns.

- "Linear" because all the unknowns have power one and are not multiplied between them.

→ How to solve systems of equations?

• Let's start with a simple example: Solve for x and y in

a) $\left.\begin{array}{l} 4x+7y=1 \\ x-2y=4 \end{array}\right\}$ b) $\left.\begin{array}{l} 4x+7y=1 \\ y=-1 \end{array}\right\}$ c) $\left.\begin{array}{l} x=2 \\ y=-1 \end{array}\right\}$

Solution:

Obvious in c), very easy in b) as well: $y=-1$, so we substitute in the first equation
$4x+7(-1)=1 \rightarrow x=2$.

a)

We can try solving by substitution. For example, the second equation gives $x=4+2y$, so we substitute this in eq. 1:
$4(4+2y)+7y=1 \rightarrow 16+8y+7y=1 \rightarrow y=-1$.

Systematic approach:

$\left.\begin{array}{ll} E1 & 4x+7y=1 \\ E2 & x-2y=4 \end{array}\right\}$ The idea is to go from a) to c), and we do it passing through b).

• First step: Eliminate x in the second equation by substracting some multiple of the first equation.

Pivot

$$\left.\begin{array}{lr} \boxed{4}x + 7y = 1 & E1 \\ x - 2y = 4 & E2 \end{array}\right\} \xrightarrow{E2' = E2 - \frac{1}{4}E1} \left.\begin{array}{lr} 4x + 7y = 1 & E1 \\ -\frac{15}{4}y = \frac{15}{4} & E2' \end{array}\right\}$$

New system: $\left.\begin{array}{l} 4x + 7y = 1 \\ y = -1 \end{array}\right\}$ This is in "triangular" or <u>echelon form</u>.

- Step two: Solve backwards, $y = -1 \to 4x + 7(-1) = 1 \Rightarrow$
$\Rightarrow x = 2$.

So solution is $\begin{array}{|l} x = 2 \\ y = -1. \end{array}$

- <u>Example 2</u>:

$$\left.\begin{array}{r} 3x_3 = 9 \\ x_1 + 5x_2 - 2x_3 = 2 \\ \frac{1}{3}x_1 + 2x_2 = 3 \end{array}\right\}$$

<u>Solution</u>:

$$\left.\begin{array}{llr} E1 & \boxed{0}\,x_1 \qquad\qquad 3x_3 & = 9 \\ E2 & \boxed{1}\,x_1 + 5x_2 - 2x_3 & = 2 \\ E3 & \boxed{\tfrac{1}{3}}\,x_1 + 2x_2 & = 3 \end{array}\right\} \xrightarrow[E3' = E3 - \frac{1/3}{0}E1 \ !!]{E2' = E2 - \frac{1}{0}E1 \ !!}$$

We cannot use equation 1, so we rearrange our equations:

$E2 \leftrightarrow E1$

New system:

$$\left.\begin{aligned} &E1 \quad 1\cdot x_1 + 5x_2 - 2x_3 = 2 \\ &E2 \quad 3x_3 = 9 \\ &E3 \quad \tfrac{1}{3}x_1 + 2x_2 = 3 \end{aligned}\right\} \longrightarrow \begin{aligned} &E3' = E3 - \frac{1/3}{1} E1 \\ &E2' = E2 \end{aligned}$$

$$\left.\begin{aligned} &E1: \quad x_1 + 5x_2 - 2x_3 = 2 \\ &E2': \quad \boxed{0}\, x_2 + 3x_3 = 9 \\ &E3': \quad \left(2 - \tfrac{5}{3}\right)x_2 + \tfrac{2}{3}x_3 = 3 - \tfrac{2}{3} \end{aligned}\right\} \longrightarrow E3'' = E3' - \frac{2-\frac{5}{3}}{0} E2' \;\; !!$$

Rearrange again $E2' \leftrightarrow E3'$

$$\left.\begin{aligned} &E1: \quad x_1 + 5x_2 - 2x_3 = 2 \\ &E2': \quad \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3 = \tfrac{7}{3} \\ &E3': \quad 3x_3 = 9 \end{aligned}\right\}$$

This is in echelon form.

$\downarrow$ Solve backwards,

$x_3 = 3$

↳ $x_2 = 1$

↳ $x_1 = 3$.

- Gauss Elimination Algorithm

Start with $n=1$.

1) Rearrange the equations so that the n-th variable in the n-th equation has a non-zero coefficient (called pivot).

2) Use the pivot to clear out the n-th variable from all the equations below the n-th one.

3) Is the system in echelon form? $\xrightarrow{\text{Yes}}$ Solve by back substitution.

$n \leftarrow n+1$

- Example: Solve for x, y, z.

$$\left.\begin{array}{llll} E1 & x + y + z & = 4 \\ E2 & x - 2y - 2z & = 7 \\ E3 & 2x - 3y + 3z & = -5 \end{array}\right\}$$

Solution:

- $n=1$: Pivot $= 1 \longrightarrow$ $E2' = E2 - \frac{1}{1} E1$

$$E3' = E3 - \frac{2}{1} E1$$

New system:
$$\left.\begin{array}{lrcrcrcr}
E1: & x & + & y & + & z & = & 4 \\
E2': & & & -3y & - & 3z & = & 3 \\
E3': & & & -5y & + & z & = & -13
\end{array}\right\}$$
Not in echelon form.

- $\underline{n=2}$: Pivot $= -3$

$$\hookrightarrow E3'' = E3' - \frac{-5}{-3}\,E2'$$

New system:
$$\left.\begin{array}{lrcrcrcr}
E1 & x & + & y & + & z & = & 4 \\
E2' & & & -3y & - & 3z & = & 3 \\
E3'' & & & & & 6z & = & -18
\end{array}\right\}$$

The system is in echelon form, so we solve backwards:

$z = -3 \longrightarrow y = 2 \longrightarrow x = 5.$ Sol: $\left|\begin{array}{l} x = 5 \\ y = 2 \\ z = -3 \end{array}\right.$

- Breakdown of Gaussian Elimination: Singular Cases:

<u>Case 1</u>	<u>Case 2</u>	<u>Case 3</u>
$\left.\begin{aligned} x + y + z &= 4 \\ 2x + 2y - 2z &= 7 \\ 2x - y - z &= 11 \end{aligned}\right\}$	$\left.\begin{aligned} x + y + z &= 4 \\ x - 2y - 2z &= 7 \\ 2x - y - z &= 10 \end{aligned}\right\}$	$\left.\begin{aligned} x + y + z &= 4 \\ x - 2y - 2z &= 7 \\ 2x - y - z &= 11 \end{aligned}\right\}$

<u>Case 1</u>: This is <u>not</u> a singular case, it simply needs reordering the equations at some point:

$$\text{Pivot} = 1 \rightarrow \left.\begin{aligned} E2' &= E2 - 2E1 \\ E3' &= E3 - 2E1 \end{aligned}\right\} \rightarrow \begin{array}{ll} E1 & x + y + z = 4 \\ E2' & -4z = -1 \\ E3' & -3y - 3z = 3 \end{array}$$

Rearrange $E3' \leftrightarrow E2'$ and the system will be in echelon form.

<u>Case 2</u>

$$\begin{array}{l} \text{Pivot} = 1 \rightarrow \\ (n=1) \end{array} \left.\begin{aligned} E2' &= E2 - 2E1 \\ E3' &= E3 - 2E1 \end{aligned}\right| \rightarrow \begin{array}{ll} E1 & x + y + z = 4 \\ E2' & -3y - 3z = 3 \\ E3': & -3y - 3z = 2 \end{array}$$

$$(n=2)\ \text{Pivot} = -3 \rightarrow E3'' = E3' - \frac{-3}{-3} E2' \rightarrow \begin{array}{ll} E1' & x + y + z = 4 \\ E2' & -3y - 3z = 3 \\ E3'' & 0 = -1 \leftarrow !! \end{array}$$

→ The system does not have a solution.

Case 3:

(n=1) Pivot = 1 → $E2' = E2 - E1$, $E3' = E3 - 2E1$ →

$$\left.\begin{array}{ll} E1 & x + y + z = 4 \\ E2' & -3y - 3z = 3 \\ E3' & -3y - 3z = 3 \end{array}\right\}$$

(n=2) Pivot = −3 → $E3'' = E3' - E2$ →

$$\left.\begin{array}{ll} E1 & x + y + z = 4 \\ E2' & -3y - 3z = 3 \\ E3'' & 0 = 0 \end{array}\right\}$$

The system is now in echelon form and it has more variables than pivots, thus there are infinitely many solutions.

Remark: These are all possible cases. Apply Gaussian Elimination and find that

1) The system is reduced to its echelon form, with as many pivots as variables → 1 solution (solve backwards).

2) An equation of the type "0 = 1" is obtained ⇒ No solutions.

3) The echelon form has less pivots than variables ⇒ → Infinitely many solutions.