MATH 115
LECTURE 20

# Gauss-Jordan Elimination

- Systems in matrix form:

we can write a system like the following one

$2x - y + z = 2$ E1

$4x + y - z = 1$ E2

$2x + 2y + 2z = 1$ E3

using the so-called augmented matrix:

$$\begin{array}{cc} R1 \\ R2 \\ R3 \end{array} \left[\begin{array}{ccc|c} \boxed{2} & -1 & 1 & 2 \\ 4 & 1 & -1 & 1 \\ 2 & 2 & 2 & 1 \end{array}\right]$$

Now, the gaussian elimination steps are row operations:

$$\begin{array}{l} R2' = R2 - 2R1 \\ R3' = R3 - R1 \end{array} \rightarrow \left[\begin{array}{ccc|c} \boxed{2} & -1 & 1 & 2 \\ 0 & \boxed{3} & -3 & -3 \\ 0 & 3 & 1 & -1 \end{array}\right] \xrightarrow{R3'' = R3' - R2'} \underbrace{\left[\begin{array}{ccc|c} \boxed{2} & -1 & 1 & 2 \\ 0 & \boxed{3} & -3 & -3 \\ 0 & 0 & \boxed{4} & 2 \end{array}\right]}_{\text{echelon form}}$$

Therefore, backwards substitution:

$$4z = 2 \rightarrow z = \frac{1}{2} \rightarrow 3y - 3\frac{1}{2} = -3 \rightarrow \boxed{y = \frac{-1}{2}} \rightarrow \dots \boxed{x = \frac{1}{2}}$$

Example: $\left.\begin{aligned} 2x_1 - x_2 + x_3 + x_4 &= 2 \\ 4x_1 + x_2 - x_3 + 2x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 - x_4 &= 1 \end{aligned}\right\}$ Find the solution/s or show there aren't any.

Sol:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 2 & -1 & 1 & 1 & 2 \\ 4 & 1 & -1 & 2 & 1 \\ 2 & 2 & 2 & -1 & 1 \end{array}\right] \end{array} \xrightarrow[R3' = R3 - R1]{R2' = R2 - 2R1} \left[\begin{array}{cccc|c} \boxed{2} & -1 & 1 & 1 & 2 \\ 0 & \boxed{3} & -3 & 0 & -3 \\ 0 & 3 & 1 & -2 & -1 \end{array}\right] \xrightarrow{R3'' = R3' - R2}$$

$$\rightarrow \left[\begin{array}{cccc|c} \boxed{2} & -1 & 1 & 1 & 2 \\ 0 & \boxed{3} & -3 & 0 & -3 \\ 0 & 0 & \boxed{4} & -2 & 2 \end{array}\right]$$

echelon form

"free variable" (no pivot in its column).

We set the free variables as parameters: $x_4 = \alpha \in \mathbb{R}$, and solve for the other variables in terms of these parameters:

$$4x_3 - 2\alpha = 2 \Rightarrow x_3 = \frac{1}{2} + \frac{1}{2}\alpha$$

$$3x_2 - 3\left(\frac{1}{2} + \frac{1}{2}\alpha\right) = -3 \Rightarrow x_2 = \frac{-1}{2} + \frac{1}{2}\alpha. \qquad (*)$$

$$2x_1 - \left(\frac{-1}{2} + \frac{1}{2}\alpha\right) + \left(\frac{1}{2} + \frac{1}{2}\alpha\right) + \alpha = 2 \Rightarrow x_1 = \frac{1}{2} - \frac{1}{2}\alpha$$

Notice that we can write the solution in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

The steps (*) are not the best way to proceed. Moreover, the echelon form that we obtain when we apply gaussian elimination is not unique. We shall see a more complete method, that avoids the part of substituting backwards.

- <u>Gauss-Jordan Elimination</u>

The method consists in applying gaussian elimination until we obtain the echelon form, and then we apply gaussian elimination upwards until we obtain the so-called "reduced echelon form": every pivot has to be 1, and all the other entries in its column are zero.

<u>Example</u>:
$$\left.\begin{aligned} x_1 - x_3 + 4x_4 &= -1 \\ x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 2x_2 + 4x_3 - 4x_4 &= 8 \end{aligned}\right\}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -1 & 4 & -1 \\ 1 & \boxed{1} & 1 & 0 & 1 \\ 2 & 2 & 4 & -4 & 8 \end{array}\right] \xrightarrow[R3' = R3 - 2R1]{R2' = R2 - R1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & -1 & 4 & -1 \\ 0 & \boxed{1} & 2 & -4 & 2 \\ 0 & 2 & 6 & -12 & 10 \end{array}\right] \xrightarrow{R3'' = R3' - 2R2'}$$

$$\to \underbrace{\left[\begin{array}{cccc|c} \boxed{1} & 0 & -1 & 4 & -1 \\ 0 & \boxed{1} & 2 & -4 & 2 \\ 0 & 0 & \boxed{2} & -4 & 6 \end{array}\right]}_{\text{echelon form.}} \xrightarrow[R1' = R1 + \frac{1}{2}R3'']{R2'' = R2' - R3''} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 2 \\ 0 & \boxed{1} & 0 & 0 & -4 \\ 0 & 0 & \boxed{2} & -4 & 6 \end{array}\right] \to$$

$$\xrightarrow{R3''' = \frac{1}{2}R_3''} \underbrace{\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 2 & 2 \\ 0 & \boxed{1} & 0 & 0 & -4 \\ 0 & 0 & \boxed{1} & -2 & 3 \end{array}\right]}_{\text{Reduced Echelon Form}}$$

free variable

Therefore,

$$x_4 = \alpha \in \mathbb{R}$$

$$x_1 = 2 - 2\alpha$$

$$x_2 = -4$$

$$x_3 = 3 + 2\alpha$$

$$\to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

<u>Exercise</u>: Consider the planes $\left.\begin{aligned} x+2y+z&=1\\ x\qquad -z&=0\end{aligned}\right\}$

Find the intersection

<u>Sol</u>:
We want to find all points $(x,y,z)$ that are in both planes, that is, the solutions to the system above.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1\\ 1 & 0 & -1 & 0\end{array}\right] \xrightarrow{R2' = R2 - R1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1\\ 0 & -2 & -2 & -1\end{array}\right] \xrightarrow{R1' = R1 + R2'}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & -2 & -2 & -1\end{array}\right] \xrightarrow{R2'' = \frac{-1}{2}R2'} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0\\ 0 & \boxed{1} & 1 & 1/2\end{array}\right]$$

free

Thus,

$$\begin{aligned} z &= \alpha \in \mathbb{R}\\ x &= \alpha\\ y &= \tfrac{1}{2} - \alpha \end{aligned} \quad \rightarrow \quad \begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} 0\\ 1/2\\ 0\end{bmatrix} + \alpha \begin{bmatrix} 1\\ -1\\ 1\end{bmatrix}$$

<u>Remark</u>: Notice that we can write the system using matrix multiplication,

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_1 \qquad - x_3 &= 0 \end{aligned} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 0 & -1 & 0 \end{array}\right]$$

with $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\downarrow$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{b}}$$

The solution is divided in two pieces:

$$\vec{x} = \underbrace{\vec{x}_p}_{\text{particular solution}} + \underbrace{\vec{x}_h}_{\text{homogeneous solution}} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\uparrow$

this is the solution to $A\vec{x} = \vec{0}$

• **Exercise:** A system $A\vec{x}=\vec{b}$ is transformed through Gauss-Jordan into reduced echelon form $R\vec{x}=\vec{d}$. Find $R$ and $\vec{d}$ if the complete solution is

(A is $4\times 4$).

$$\vec{x}=\begin{bmatrix}4\\0\\1\\0\end{bmatrix}+\alpha\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+\beta\begin{bmatrix}-1\\0\\2\\1\end{bmatrix}$$

Solution:

Two parameters $\rightarrow$ Two free variables. Which ones?

$$\left.\begin{aligned} x_1&=4+\alpha-\beta\\ \rightarrow x_2&=\alpha\\ x_3&=1+2\beta\\ \rightarrow x_4&=\beta \end{aligned}\right\}\rightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \begin{bmatrix}\boxed{1}&-1&0&1\\0&0&\boxed{1}&-2\\0&0&0&0\\0&0&0&0\end{bmatrix} \quad \vec{d}=\begin{bmatrix}4\\1\\0\\0\end{bmatrix}$$

**Exercise:** Find the parabola $y=ax+bx^2+c$ that goes through the points $(0,1)$, $(1,3)$, $(2,7)$.

Sol

$$\left.\begin{aligned}1&=c\\3&=a+b+c\\7&=2a+4b+c\end{aligned}\right\}\rightarrow \begin{aligned}3&=a+b+1\\7&=2a+4b+1\end{aligned}\rightarrow\begin{bmatrix}1&1\\2&4\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}\rightarrow$$

$$\rightarrow\left[\begin{array}{cc|c}1&1&2\\0&2&2\end{array}\right]\rightarrow \begin{aligned}&\boxed{b=1}\\ &a=2-1=\boxed{1=a}\end{aligned}.$$