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Problem 1

1) $y' = -5y \Rightarrow \boxed{y(t) = c_1 e^{-5t}}$

2) $xy' + 3x^3 y = 0$

$$x\frac{dy}{dt} = -3x^3 y$$

$$\int \frac{dy}{dt} = \int (-3x^2) y$$

$$\boxed{y(t) = c_1 e^{(-3x^2)t}}$$

3) $y'' = -4y$

$r^2 = -4$

$r = \pm 2i \Rightarrow y(t) = c_1 e^{+2it} + c_2 e^{-2it}$

Recall: $\sin(t) = \frac{e^{it} - e^{-it}}{2i}$, $\cos(t) = \frac{e^{it} + e^{-it}}{2}$

$$\cos(2t) + i\sin(2t) = \frac{e^{2it} + e^{-2it}}{2} + \frac{e^{2it} - e^{-2it}}{2} = \boxed{e^{2it}}$$

$$\cos(2t) - i\sin(2t) = \frac{e^{2it} + e^{-2it}}{2} - \frac{e^{2it} - e^{-2it}}{2} = \boxed{e^{-2it}}$$

$$y(t) = c_1(\cos(2t) + i\sin(2t)) + c_2(\cos(2t) - i\sin(2t))$$

$$y(t) = (c_1 - c_2)\, i \sin(2t) + (c_1 + c_2)\cos(2t)$$

$$y(t) = c_3 i \sin(2t) + c_4 \cos(2t)$$

both c_3 & c_4 are complex. in our final solution, we only care about the real valued coeffs.!

on your own slip :)

$$\boxed{y(t) = c_3 \sin(2t) + c_4 \cos(2t)}$$

4) $y'' = 4y$

$r^2 = 4$

$r = \pm 2 \Rightarrow y(t) = c_1 e^{2t} + c_2 e^{-2t}$

Recall: $\sinh(t) = \frac{e^{t} - e^{-t}}{2}$, $\cosh(t) = \frac{e^{t} + e^{-t}}{2}$

$$\sinh(2t) + \cosh(2t) = e^{2t}$$

$$\cosh(2t) - \sinh(2t) = e^{-2t}$$

$\Downarrow$

$$y(t) = c_1\left(\sinh(2t) + \cosh(2t)\right) + c_2\left(\cosh(2t) - \sinh(2t)\right)$$

$$= (c_1 - c_2)\sinh(2t) + (c_1 + c_2)\cosh(2t)$$

$$\boxed{y(t) = c_3 \sinh(2t) + c_4 \cosh(2t)}$$

5) $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$\underline{r = 2}$

$\hookrightarrow \boxed{y(t) = c_1 e^{2t} + c_2 t e^{2t}}$

Problem 2

a) $\omega = c_1 v_1 + \dots + c_n v_n$

$$c_i = \frac{\langle \omega, v_i \rangle}{\langle v_i, v_i \rangle} = \frac{\cancel{(c_1 v_1)}^{0} v_i + \cancel{(c_2 v_2)}^{0} v_i + \dots + (c_i v_i) v_i}{v_i \cdot v_i}$$

$*$ $\langle v_i, v_j \rangle = 0$ for $i \neq j$

$$= c_i \frac{v_i \cancel{v_i}^{1}}{\cancel{v_i v_i}}$$

$$\boxed{= c_i} \checkmark$$

b) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & -1 & 5 \end{array}\right] \overset{RREF}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & \frac{9}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}$

Prob 3 (This question uses many techniques that you may not have been taught yet)

$\langle u,u\rangle = 9 \Rightarrow |\vec{u}| = 3$

$\langle v,v\rangle = 1 \Rightarrow |\vec{v}| = 1$

$\langle w,w\rangle = 38 \Rightarrow |\vec{w}| = \sqrt{38}$

$\langle u,v\rangle = 0$

$\langle u,w\rangle = 6 = |u||w|\cos\theta = 3\cdot\sqrt{38}\cos\theta \to 71.1^\circ$

$\langle w,v\rangle = 3 = |w||v|\cos\theta = \sqrt{38}\cos\theta \to 60.9^\circ$

a) $\hat{u} = \frac{u}{\|u\|}$

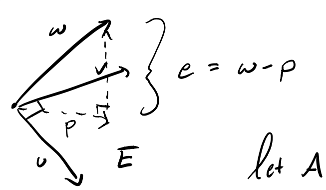
b) $u\cdot(v+w) = u\cdot v + u\cdot w$

$\boxed{= 0 + 6}$

c) $\|v+w\|^2 = \|v\|^2 + \|w\|^2 + 2\|v\|\|w\|$

$= 1 + 38 + 3(2)$

$\boxed{\|v+w\| = \sqrt{45} = 3\sqrt{5}}$

d) 

$p = A\hat{x}$

let A represent the matrix that contains vectors u, v that span E $\quad A = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix}$

$A^T(w-p) = 0$

$A^T(w - A\hat{x}) = 0$

$A^Tw = A^TA\hat{x}$

$\hat{x} = (A^TA)^{-1}A^Tw$

$p = A(\underbrace{\cancel{A^TA}}_{I \text{ (orthogonal)}})^{-1}A^Tw = AA^Tw = \boxed{u^Twu + v^Twv}$

e) $e = w - p$

$e = w - AA^Tw$

$e = w - u^Twu - v^Twv$

$\boxed{\hat{e} = \frac{e}{\|e\|}}$

f) Omitted.

Problem 4

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\Downarrow$$

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(order of derivatives doesn't matter)

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}} \checkmark$$

$$u(x,t) = f(x+t)$$

$$\frac{\partial}{\partial t}\left(f(x+t)\right) \stackrel{?}{=} \frac{\partial}{\partial x}\left(f(x+t)\right)$$

$$\left.\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}[f(x+t)] = f'(x+t)\cdot 1 \quad \text{(chain rule)} \\ \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}[f(x+t)] = f'(x+t)\cdot 1 \quad \text{(chain rule)} \end{aligned}\right\} \text{same} \checkmark$$

Problem 5

a) $$\operatorname{div}(f\vec{F}) = \nabla f \cdot \vec{F} + f \operatorname{div}\vec{F}$$

Plug in example $\underline{f(x,y,z)}$ & $\underline{\vec{F}}$ and just show that this relationship holds.

b) $$\int_\Omega f \cdot \Delta g = \int_{\partial\Omega} f \frac{\partial g}{\partial n} - \int_\Omega \nabla f \cdot \nabla g$$

$$\nabla(\nabla g \cdot f) = \nabla^2 g f + \nabla g \nabla f$$

$$\int_\Omega \nabla(\nabla g \cdot f) = \int_\Omega \nabla^2 g f + \int_\Omega \nabla g \nabla f$$

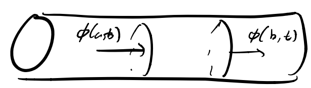
$$\int_{\partial\Omega} \nabla g \cdot \hat{n} \cdot f = \int_\Omega \nabla^2 g f + \int_\Omega \nabla g \nabla f$$

$$\int_\Omega \nabla^2 g f = \int_{\partial\Omega} \nabla g \cdot \hat{n} \cdot f - \int_\Omega \nabla g \nabla f$$

Problem 6

1.2.1

a)



$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$$

Method 1

* If heat energy flux is INCREASING, then heat flows out to the right than in from the left $\Rightarrow$ thermal energy is decreasing!

$$\frac{\partial e}{\partial t} = \underbrace{\lim_{\Delta x \to 0} \frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x}}_{\text{def. of derivative}} + \cancel{Q(x,t)}^{0} \quad (1.2.2)$$

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}}$$

Method 2

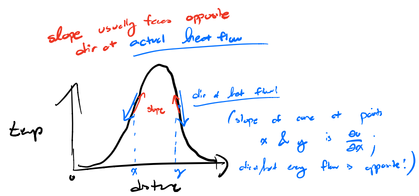
$$\underbrace{\frac{d}{dt}\int_a^b e\,dx}_{\text{rate of heat energy change}} = \underbrace{\phi(a,t) - \phi(b,t)}_{\text{heat flux through sides}} + \cancel{\int_a^b Q\,dx}^{0} \quad (1.2.4)$$

$$\int_a^b \frac{\partial e}{\partial t}\,dx = -\int_a^b \frac{\partial \phi}{\partial t}\,dx$$

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial t}}$$

b) $\phi = -K_0 \dfrac{\partial u}{\partial x}$

a. Heat flows from hotter region to colder region



c) $$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}$$

If more chemical flows out than in, then the overall chemical concn decreases!

If more chemical flows in than out THEN the overall chemical concn increases!

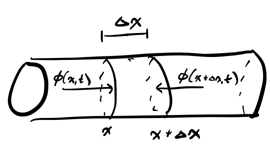
(similar explanation from a)

d) Also similar explanation to b

Problem 7

1.2.2

a)



$$\frac{\partial}{\partial t}\left[e(x,t)\Delta x \cancel{A}\right] = \phi(x,t)\cancel{A} - \phi(x+\Delta x,t)\cancel{A}$$

$$\frac{\partial}{\partial t}\left[e(x,t)\right] = \frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x}$$

take lim $\Delta x \to 0$ (definition of derivative

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$$

Recall:

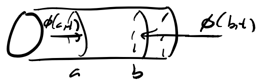
Fourier's law: $\phi = -K_0 \dfrac{\partial u}{\partial x}$

$e(x,t) = u(x,t)c\rho$

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{\partial u}{\partial t} = \left(\frac{K_0}{c\rho}\right)\frac{\partial^2 u}{\partial x^2}}$$

b)



$$\frac{d}{dt}\int_a^b e\,A\,dx = \phi(a,t)A - \phi(b,t)A$$

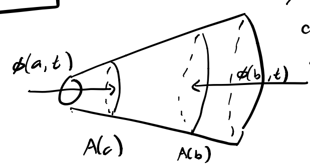
$$= -A\int_a^b \frac{\partial \phi}{\partial x}\,dx$$

$$\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x}$$

$$\boxed{\frac{\partial u}{\partial t} = \left(\frac{k_0}{c\rho}\right)\frac{\partial^2 u}{\partial x^2}}$$

Problem 8

1.2.3



let $A(x)$ represent the cross sectional area as a fcn of x

$$\frac{d}{dt}\int_a^b e(x,t)A(x)\,dx = \phi(a,t)A(a) - \phi(b,t)A(b)$$

$$= -\int_a^b \frac{\partial}{\partial x}\left[\phi(x,t)A(x)\right]dx$$

$$\frac{\partial e}{\partial t}A(x) = -\frac{\partial}{\partial x}\left[\phi A(x)\right]$$

Recall:

Fourier's law: $\phi = -K_0\frac{\partial u}{\partial x}$

$e(x,t) = u(x,t)c\rho$

$$\frac{\partial u}{\partial t}c\rho A(x) = K_0\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}A(x)\right]$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{1}{A(x)}\frac{K_0}{c\rho}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}A(x)\right]}$$

Problem 9:

This derivation basically uses the same ideas in problem 7

Problem 10

Read.