MATH 241 LECTURE 1: Review of prerequisite concepts.

- Syllabus
- Introduction: → Midterm 1 ↗ Heat and Laplace equations.
  ↳ Method: Separation of variables.

  → Midterm 2 ↗ Wave equation
  ↳ Method: Fourier series.

  → Final: Higher dimensions and Fourier transform.
  (cumulative).

Example of a PDE: $u_{xx}(x,y) = 0$. $\left(u_{xx}(x,y) := \frac{\partial^2}{\partial x^2}(u(x,y)) \right)$.

Solution?

↳ Think first of the ODE $f''(x) = 0 \rightsquigarrow f(x) = c_1 x + c_2$.
(c_1, c_2 constants)

Analogously,

$u_{xx}(x,y) = 0 \rightsquigarrow u(x,y) = c_1(y)x + c_2(y)$

Moral: In PDE the solutions have undetermined functions playing the role of the constants in ODE.

↳ conditions will be needed to find those functions.

→ It is clear that to solve PDEs, we need to know how to solve ODEs.

- Review of ODEs

1) Separable ODEs: $u'(x) = f(x)\,g(u)$ for given functions f, g.

Solution:
$$\frac{du}{dx} = f(x)\,g(u) \Rightarrow \int \frac{du}{g(u)} = \int f(x)\,dx + C$$

Example:
$$\left.\begin{array}{l} y'(x) = x y^2 \\ y(0) = 1 \end{array}\right\}$$

Sol:
$$\frac{dy}{y^2} = x\,dx \Rightarrow \frac{-1}{y(x)} = \frac{x^2}{2} + C \underset{y(0)=1}{\Rightarrow} -1 = 0 + C.$$

Thus, $y(x) = \dfrac{-2}{x^2-2}$.

Example: $u'(t) + (u(t))^2 \cos(t) = 0$

Sol:
$$\int \frac{du}{u^2} = -\int \cos(t)\,dt \Rightarrow \frac{-1}{u(t)} = -\sin(t) + C \Rightarrow u(t) = \frac{1}{\sin(t) - C}.$$

2) <u>Linear ODEs</u> (very important for this course).

2.1) 1st order linear ODEs

2.1.1) With constant coefficients: $\| u'(x) + b\,u(x) = h(x) \|$ $(b \in \mathbb{R})$

Solution:

Let's do first a particular example: $u'(x) + 2u(x) = 0$.

Multiply by the integration factor e^{2x}:

$e^{2x} u' + 2e^{2x} u(x) = 0$ and notice that

$e^{2x} u'(x) + 2e^{2x} u(x) = \frac{d}{dx}\left(e^{2x} u(x)\right)$ therefore the equation is reduced to

$$\frac{d}{dx}\left(e^{2x} u(x)\right) = 0 \Rightarrow e^{2x} u(x) = c \Rightarrow u(x) = c\,e^{-2x} /\!/.$$

→ General case: $u'(x) + b\,u(x) = h(x)$.

Use the integration factor e^{bx},

$$e^{bx} u'(x) + b e^{bx} u(x) = \frac{d}{dx}\left(e^{bx} u(x)\right) = e^{bx} h(x) \Rightarrow$$

$$\Rightarrow e^{bx} u(x) = \int e^{bs} h(s)\,ds + c \Rightarrow$$

$$\Rightarrow \boxed{u(x) = e^{-bx} \int e^{bs} h(s)\,ds + e^{-bx} c}$$

2.1.2) General coefficients: $u'(x) + p(x)u(x) = g(x)$.

"Same" method: multiply by $e^{\int p(x)dx}$:

$$e^{\int p(x)dx} u'(x) + e^{\int p(x)dx} p(x)u(x) = \frac{d}{dx}\left(e^{\int p(x)dx} u(x)\right) = e^{\int p(x)ds} g(x)$$

$$\Rightarrow \left\| u(x) = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} g(x)dx + C\right)\right\|$$

- Example: $\left.\begin{array}{l} u'(x) + \frac{1}{x}u(x) = x^2 \\ u(1) = 1 \end{array}\right\}$

Sol: $e^{\int \frac{1}{x}dx} = e^{\log(x)} = x$,

$$x u'(x) + u(x) = \frac{d}{dx}(x u(x)) = x^3 \Rightarrow x u(x) = \frac{x^4}{4} + C \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow u(x) = \dfrac{x^3}{4} + \dfrac{C}{x}. \\ u(1) = 1 = \dfrac{1}{4} + C \Rightarrow C = \dfrac{3}{4} \end{array}\right\} \Rightarrow u(x) = \frac{1}{4}\left(x^3 + \frac{3}{x}\right).$$

2.2) 2^{nd} order linear ODEs

We will only consider homogeneous ODEs with constant coefficients in this case,

$$\boxed{u''(x) + b\,u'(x) + c\,u(x) = 0} \quad (b, c \in \mathbb{R})$$

Solution: We study the roots of the characteristic polynomial.

$$r^2 + br + c = 0 \Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Three cases:

① Two real and distinct numbers: r_1, r_2. Then,

$$u(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

② Two complex roots (conjugates): real part $\frac{-b}{2}$.

imaginary part $\frac{\sqrt{4c - b^2}}{2}$.

Denoting $\omega = \frac{\sqrt{4c-b^2}}{2}$, the solution is

$$u(x) = e^{-b\frac{x}{2}}\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right).$$

③ Two real repeated roots: $r_1 = r_2 = \frac{-b}{2}$.

Solution: $u(x) = c_1 e^{-b\frac{x}{2}} + c_2 \underline{x}\, e^{-bx/2}$.

Example: $$\left.\begin{aligned} &u''(x) - 9u(x) = 0 \\ &u(0) = 2 \\ &u'(0) = -1 \end{aligned}\right\}$$

Sol: $r^2 - 9 = 0 \to r = \pm 3 \to u(x) = c_1 e^{3x} + c_2 e^{-3x}$.

Initial data $\to$ $$\left.\begin{aligned} &u(0) = c_1 + c_2 = 2 \\ &u'(0) = \left(3c_1 e^{3x} - 3c_2 e^{-3x}\right)\Big|_{x=0} = 3c_1 - 3c_2 = -1 \end{aligned}\right\} \to$$

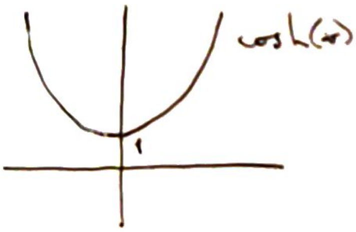
$$\to \left.\begin{aligned} c_1 + c_2 &= 2 \\ 3c_1 - 3c_2 &= -1 \end{aligned}\right| \to 6c_1 = 5 \to c_1 = \frac{5}{6},\ c_2 = 2 - \frac{5}{6} = \frac{7}{6} \,/\!/.$$

Remark: The solution can also be written as

$u(x) = c_3 \cosh(3x) + c_4 \sinh(3x)$, since (they span the same vector space of functions).

$$\left\| \begin{aligned} \cosh(x) &= \frac{e^x + e^{-x}}{2} \\ \sinh(x) &= \frac{e^x - e^{-x}}{2} \end{aligned} \right\|$$

$\hookrightarrow$ $\begin{aligned} (\cosh(x))' &= \sinh(x) \\ (\sinh(x))' &= \cosh(x) \end{aligned}$, $\cosh^2(x) - \sinh^2(x) = 1$, $\begin{aligned} &\cosh(0) = 1, \\ &\sinh(0) = 0. \end{aligned}$



- ## Review of vector calculus

In ODEs we have functions of one variable and its derivatives.

In PDEs we have functions of several variables and its partial derivatives.

Let $f(x,y,z)$,

→ Gradient: $\nabla f = (f_x, f_y, f_z)$.

→ Directional derivative: $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$
(with $|\vec{u}|=1$)

↳ Thus $\vec{u} = \dfrac{\nabla f}{|\nabla f|}$ is the direction in which f increases fastest ($\theta = 0 \Rightarrow \cos\theta = 1$).

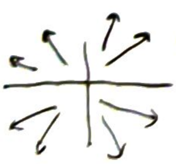
→ Laplacian: $\Delta f = f_{xx} + f_{yy} + f_{zz}$.

→ Divergence: If we have a vector field $\vec{F}(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$,

then $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \partial_x f_1 + \partial_y f_2 + \partial_z f_3$.

A positive divergence means there is a source for the flow of $\vec{F}$.

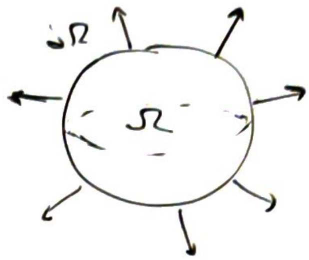
Example: $f(x,y) = x^2 + y^2 \rightarrow \nabla f(x,y) = (2x, 2y)$

 $\nabla f \xrightarrow{\nabla \cdot \nabla f}$ $\nabla \cdot \nabla f = \Delta f = 2 + 2 = 4 > 0$.

- A natural way of understanding the divergence of a field as sources and sinks is through the "divergence theorem":

<u>Divergence Theorem</u>: Let $\Omega \subset \mathbb{R}^3$ a bounded region, $\partial\Omega$ a smooth boundary and $\vec{F}$ a vector field in $\mathbb{R}^3$. Then,

$$\iiint_{\Omega} \operatorname{div} \vec{F} = \underbrace{\iint_{\partial\Omega} \vec{F}\cdot\vec{n}}_{\substack{\text{Flux of } \vec{F} \text{ through} \\ \text{the boundary}}}$$



($\vec{n}$ = unit outward normal vector to $\partial\Omega$).

<u>Example</u>: $\vec{F}(x,y,z) = (x,y,z)$. Compute the flux of $\vec{F}$ through the unit sphere.

<u>Sol.</u> $$\iint_{\partial\Omega} \vec{F}\cdot\vec{n} = \iiint_{\Omega} \operatorname{div}\vec{F} = \iiint_{\text{unit ball}} 3 = 3\cdot \mathrm{Vol}(\text{unit ball}) = 3\,\frac{4\pi}{3} = 4\pi.$$

→ Remember integration by parts: $\displaystyle\int_a^b f g' = f g\Big|_a^b - \int_a^b f' g.$

↳ Proof: $(fg)' = f'g + fg' \Rightarrow$

$\Rightarrow \displaystyle\int_a^b (fg)' = fg\Big|_a^b = \int_a^b f'g + \int_a^b fg'$ (rearrange).

→ The higher dimensional version of integration by parts also holds

$$\int_{\Omega} f \Delta g = \int_{\partial\Omega} f \nabla g \cdot \vec{n} - \int_{\Omega} \nabla f \cdot \nabla g \qquad \left(\begin{array}{l}\text{see} \\ \text{homework 1}\end{array}\right).$$

<u>Remark</u>: We will use the simple $\int$ symbol for all integrals.