MATH 241 LECTURE 2: Derivation of the heat equation.

(~ Section 1.2)

In this course we will study mostly three PDEs:

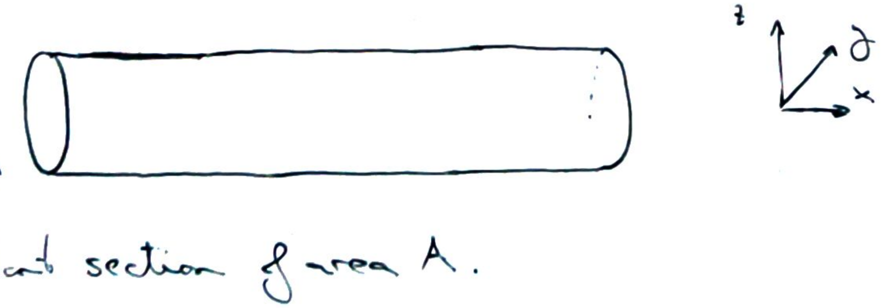
heat, wave and Laplace equations.

- Today we are going to obtain the heat equation from physics. Mathematically, the equation is the same for diffusion problems, and indeed it is sometimes called the diffusion equation.

As opposed to the book, we will first obtain the equation using this second physical picture.

- <u>The diffusion equation</u> (1d)

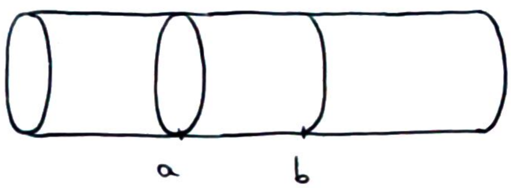
Imagine a closed pipe filled with water at rest, which contains some chemical dissolved:



It is known that the chemical substance tends to distribute uniformly among the water. Given the distribution of the chemical at a certain time, we want to find a PDE that governs its evolution.

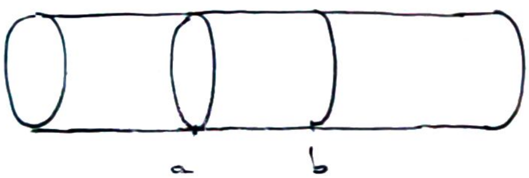
Let's denote $\rho(x,t) \equiv$ density of the chemical at position x and time t,

and consider an arbitrary region of the pipe, i.e., choose two points a and b:



time t
Section A
Density $\rho(x,t)$.

Now, we look at the same region after a short time $h>0$:



time $t+h$
Section A
Density $\rho(x,t+h)$.

1| The $\boxed{\text{conservation of mass}}$ applied to the chemical in the region $[a,b]$ tells us that

" Mass in $[a,b]$ at time $t+h$ = Mass in $[a,b]$ at time t + Mass that has come in − Mass that has come out."

① ② ③ ④

Let's model mathematically the terms ① to ④ :

①,② : In terms of the density, we can easily write the mass in $[a,b]$ at a certain instant of time.

$$\text{density} \sim \frac{\text{mass}}{\text{volume}} \Rightarrow \begin{array}{l}\text{Mass in } [a,b] \\ \text{at time } t\end{array} = \int_{\substack{\text{Volume} \\ \text{in } [a,b]}} \rho(x,y,z,t)\, dV,$$

Here, since we are assuming that the density is constant among each section (i.e., only depends on x) and that the area of the section is a constant A, we have that

$$\begin{array}{l}\text{Mass in } [a,b] \\ \text{at time } t\end{array} = \int_a^b \rho(x,t)\, A\, dx = A\int_a^b \rho(x,t)\, dx.$$

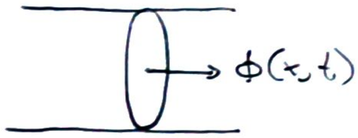
Similarly,

$$\begin{array}{l}\text{Mass in } [a,b] \\ \text{at time } t+h\end{array} = A\int_a^b \rho(x,t+h)\, dx.$$

③,④ : The chemical can only come in or out of the region through the section at $x=a$ and $x=b$.

Let's define the **flux** through a surface :

$\phi(x,t) \equiv$ amount (of mass*) per unit time flowing to the right per unit surface area.
(Flux)



→ Notice the sign convention:

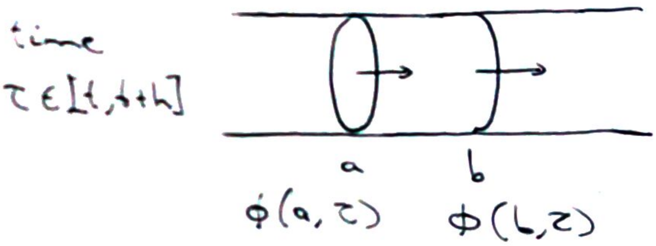
$\phi(x,t) > 0 \Rightarrow$ Flow going to the right.

$\phi(x,t) < 0 \Rightarrow$ Flow crossing to the left.

(* It is defined analogously for other physical quantities).

In summary,

$$\text{mass flux} \approx \frac{\text{mass}}{\text{time} \times \text{area}} \text{ and thus}$$



Mass that has "come in" : $\displaystyle\int_t^{t+h} \int_{\substack{\text{section}\\ \text{at } a}} \phi(a,y,z,\tau)\, dA\, d\tau$

Mass that has "come out" : $\displaystyle\int_t^{t+h} \int_{\substack{\text{section}\\ \text{at } b}} \phi(b,y,z,\tau)\, dA\, d\tau.$

Again, since our problem is one-dimensional, we can write this as follows

$$\int_t^{t+h} \int_{\substack{\text{section}\\ \text{at } a}} \phi(a,\tau)\, dA\, d\tau = \left(\text{area of section at } a\right) \int_t^{t+h} \phi(a,\tau)\, d\tau = A \int_t^{t+h} \phi(a,\tau)\, d\tau .$$

and analogously at $x=b$.

- In conclusion, the conservation of mass is

$$A \int_a^b \rho(x,t+h)\, dx = A \int_a^b \rho(x,t)\, dx + A \int_t^{t+h} \phi(a,\tau)\, d\tau - A \int_t^{t+h} \phi(b,\tau)\, d\tau .$$

- We want to obtain a PDE (partial <u>differential</u> equation), so we will get rid of the integrals.

First, since we took $h>0$ we can divide by h,

$$\int_a^b \frac{\rho(x,t+h) - \rho(x,t)}{h}\, dx = \frac{1}{h} \int_t^{t+h} \phi(a,\tau)\, d\tau - \frac{1}{h} \int_t^{t+h} \phi(b,\tau)\, d\tau,$$

and since it was arbitrary, we take the limit as $h \to 0$:

$$\int_a^b \frac{\partial}{\partial t}\big(\rho(x,t)\big)\, dx = \phi(a,t) - \phi(b,t),$$

where we have used the definition of partial derivative and the fundamental theorem of calculus.

Now, we can write $\phi(a,t) - \phi(b,t) = -\int_a^b \frac{\partial \phi}{\partial x}(x,t)\,dx$, so

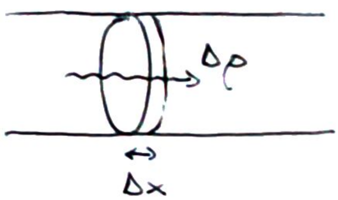
$$\int_a^b \rho_t(x,t)\,dx = -\int_a^b \phi_x(x,t)\,dx,$$

and since the interval $[a,b]$ is arbitrary as well, it must hold that

$$\boxed{\rho_t(x,t) = -\phi_x(x,t)}$$

2.) $\boxed{\text{Modelling}}$ the flux $\phi(x,t)$

It seems natural to think that the chemical will move (diffuse) from regions of higher concentration to regions where there is less:



$$\left(\frac{\Delta\rho}{\Delta x} \simeq \rho_x \right)$$

The higher the density difference $\Delta\rho$ accross the section Δx, the greater the flux is. A good assumption, which has been experimentally tested, is to consider the flux to be proportional (linear) with the derivative:

$$\boxed{\Phi(x,t) = -k(x)\,\frac{\partial \rho}{\partial x}(x,t)}$$

Fick's law of diffusion.

$k(x) \equiv$ diffusivity

Many times the diffusivity is assumed to be constant, $k(x) = k$, and then we finally obtain the diffusion equation:

$$\boxed{\rho_t = k\,\rho_{xx}}$$

Diffusion equation.

- The heat equation (1d)

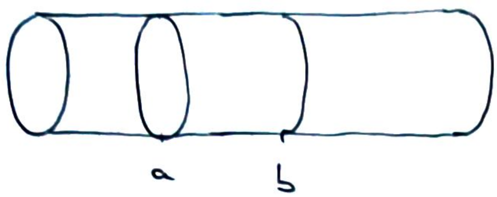
Now we want to obtain a PDE for the following physical phenomenon: consider a rod with a certain temperature distribution at a given time. We want to know how the temperature distribution changes in time.

As before, we proceed first from a fundamental physics principle: the conservation of energy.

1.1 Conservation of (thermal) energy.

This time we will write it directly in the "rate of change" form.

Situation:



→ Only dependence in x.

→ Constant area section A.

→ Lateral surfaces are insulated.

→ There can be some heat generated inside.

Then, the energy conservation states that:

rate of change of (heat) energy in time	=	energy flowing into $[a,b]$	−	energy flowing out of $[a,b]$	+	heat generated per unit time
①		②		③		④

<u>Remark</u>: Notice that we could have written the mass conservation principle in the "rate of change" form.

• Let's define a <u>heat energy density</u>: $e(x,t)$.

Then, as we did before, we obtain that

① $= \frac{d}{dt} \int_{\substack{\text{Volume} \\ \text{in } [a,b]}} e(x,y,z,t)\, dV = A \frac{d}{dt} \int_a^b e(x,t)\, dx$

(hypotheses applied)

②,③ analogous deffinition for heat flux $\left(\frac{\text{thermal energy}}{\text{time} \times \text{area}} \right)$

$$\hookrightarrow = \int_{\substack{\text{section} \\ a}} \phi(a,y,z,t)\, dA - \int_{\substack{\text{section} \\ b}} \phi(b,y,z,t)\, dA = A\left(\phi(a,t) - \phi(b,t)\right) =$$

$$= -A \int_a^b \phi_x(x,t)\, dx .$$

④ This term is new. We denote $Q(x,y,z,t)$ the heat energy per volume generated per time. so

$$\hookrightarrow = \int_{\substack{\text{Volume} \\ \text{in } [a,b]}} Q(x,y,z,t)\, dV = A \int_a^b Q(x,t)\, dx .$$

→ The integral conservation law reads as follows:

$$\frac{d}{dt} \int_a^b e(x,t)\, dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\, dx ,$$

so in differential form it is

$$\boxed{e_t = -\phi_x + Q}$$

(PDE? Which is the unknown?
$\downarrow$
we need models for e and ϕ.).

2.) Modelling (constitutive laws).

We will write everything in terms of the temperature of the rod:

- $e(x,t) = c(x)\,\rho(x)\,u(x,t)$

  u: temperature; c: specific heat; ρ: (mass) density

> Roughly, the temperature measures how fast molecules move, the density measures "how many" molecules are there, and the specific heat measures how "strong" the molecules are.

- Flux: $\boxed{\phi(x,t) = -K(x)\,\frac{\partial u}{\partial x}(x,t)}$ Fourier's law of heat conduction.

  K: thermal conductivity.

(same interpretation than Fick's law).

- $Q(x,t)$ is usually a given function (data).

• When c, ρ, K are constants, and $Q = 0$, we obtain the heat equation:

$$\boxed{u_t = k\, u_{xx}}$$ Heat equation

(k is called here thermal diffusivity, $k = \frac{K}{c\rho}$)