MATH 241 LECTURE 4 : Method of Separation of Variables I.

(~ Chapter 2)

Summary so far: we have obtained the heat equation from physical principle, then we added boundary condition and solved the equilibrium problem.

- Goal now: To solve the (time-dependent) heat equation.

<u>Remark</u>: The <u>homogeneous</u> heat equation $u_t = k u_{xx}$ is <u>linear</u>, and thus it satisfies the <u>Principle of Superposition</u>:

"If u_1, u_2 solve the a linear homogeneous equation, then any linear combination $c_1 u_1 + c_2 u_2$ also solves it".

<u>Ex</u>)
$$\left.\begin{aligned} u_{1_t} &= k u_{1_{xx}} \\ u_{2_t} &= k u_{2_{xx}} \end{aligned}\right\} \Rightarrow (c_1 u_1 + c_2 u_2)_t = c_1 u_{1_t} + c_2 u_{2_t} = c_1 k u_{1_{xx}} + c_2 k u_{2_{xx}} =$$
$$= k(c_1 u_1 + c_2 u_2)_{xx} //.$$

[Section 2.3] Heat equation with zero temperature at finite ends.

We will first solve the following problem:

$$\left.\begin{array}{lll} u_t = k u_{xx}, & 0<x<L,\ t>0 & \text{PDE} \\ u(0,t)=0 & & \\ & \text{BC} & \\ u(L,t)=0 & & \\ u(x,0)=f(x) & \text{IC} & \end{array}\right\}$$

linear and homogeneous. ("including BC).

• Method of Separation of Variables

Key point: We look for solution that can be written in the form $\boxed{u(x,t) = \phi(x)\,G(t)}$

Consequences:

$$u_t = k u_{xx} \Rightarrow \frac{\partial}{\partial t}\Big(\phi(x)G(t)\Big) = k\frac{\partial^2}{\partial x^2}\Big(\phi(x)G(t)\Big) \Rightarrow$$

$$\Rightarrow \phi(x)\,G'(t) = k\,\phi''(x)\,G(t) \Rightarrow$$

$$\Rightarrow \boxed{\frac{1}{k}\,\frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)}}$$

Question: Notice that $\frac{1}{k}\frac{G'(t)}{G(t)}$ only depends on t, while $\frac{\phi''(x)}{\phi(x)}$ only depends on x. How is then possible that they are equal?

→ The conclusion is that both terms have to be constant, the same constant:

$$\frac{1}{k}\frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} = \underbrace{-\lambda}_{\text{called the "separation constant"}}. \quad \text{(we choose } -\lambda \text{ for later convenience).}$$

→ Therefore, we have to ODEs problems to solve:

① $G'(t) = -\lambda k G(t)$

② $\phi''(x) = -\lambda \phi(x)$

In summary, so far the steps are:

1. Let $u(x,t) = G(t)\phi(x)$.
2. Use the PDE to find an ODE for $G(t)$ and another for $\phi(x)$.
   (depending on the separation constant $-\lambda$).

3. Time-dependent ODE: $G(t)$

$G'(t) = -\lambda k G(t)$ First-order linear homogeneous ODE with constant coefficients.

$$\boxed{G(t) = c e^{-\lambda k t}}$$ (We will come back to this).

4. Boundary Value Problem: $\phi(x)$

$\phi''(x) = -\lambda \phi(x)$

In this step, we include the BC of the problem (notice, however, that we are not using the initial condition yet).

We had that $\left.\begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array}\right\}$, so in terms of $G, \phi \rightarrow$

$u(0,t) = \phi(0) G(t) = 0 \Longleftrightarrow$ $\phi(0) = 0$ or $G(t) = 0$ for all t (trivial sol.)

$u(L,t) = \phi(L) G(t) = 0 \Rightarrow$ $\phi(L) = 0$ or $G(t) = 0$ for all t. (trivial sol.)

Therefore, the BVP to solve is:

$$\left.\begin{aligned} &\phi''(x)+\lambda\phi(x)=0\\ &\phi(0)=0\\ &\phi(L)=0 \end{aligned}\right\} \text{BVP}$$

The non-zero solutions are called eigenfunctions, with λ the corresponding eigenvalue.

We need to distinguish four cases:

1) $\lambda=\alpha^2>0 \rightarrow r=\pm i\sqrt{\lambda}=\pm\alpha i$

2) $\lambda=0 \rightarrow r=0$ (double)

3) $\lambda=-\alpha^2<0 \rightarrow r=\pm\sqrt{\lambda}=\pm\alpha$

4) λ complex (we will not consider this: we will prove later that λ has to be real).

• Finding the eigenvalues and eigenfunctions:

1) $\underline{\lambda=\alpha^2>0}$

Then, characteristic polynomial is $r^2+\alpha^2=0 \rightarrow r=\pm\alpha i$, so

$\phi(x)=c_1\cos(\alpha x)+c_2\sin(\alpha x)$.

Imposing the boundary conditions,

$$\left.\begin{aligned} &\phi(0)=0=c_1\\ &\phi(L)=0=\cancel{c_1}\cos(\alpha L)+c_2\sin(\alpha L) \end{aligned}\right\} \rightarrow \begin{cases} c_2=0 \text{ (then } \phi\equiv 0)\\ \underbrace{\sin(\alpha L)=0}_{\hookrightarrow}. \end{cases}$$

$\sin(\alpha L) = 0 \leftrightarrow \alpha L = n\pi$ with $n = 1, 2, \ldots \leftrightarrow$

$$\leftrightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

The corresponding eigenfunction is $\phi(x) = c_2 \sin\left(\frac{n\pi x}{L}\right)$

2) $\underline{\lambda = 0}$

$$\left.\begin{array}{l} \phi''(x) = 0 \\ \phi(0) = \phi(L) = 0 \end{array}\right\} \to \phi(x) = c_1 x + c_2$$

$$\left.\begin{array}{l} \phi(0) = c_2 = 0 \\ \phi(L) = c_1 L + c_2 = 0 \end{array}\right| \to c_1 = c_2 = 0 \to \phi \equiv 0$$

So $\lambda = 0$ is not an eigenvalue.

3) $\underline{\lambda = -\alpha^2 < 0}$

$$\phi''(x) - \alpha^2 \phi(x) = 0 \leadsto r^2 = \alpha^2 \to r = \pm\alpha \to \phi(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

which can also be written as

$$\phi(x) = c_3 \cosh(x) + c_4 \sinh(x).$$

$$\left.\begin{array}{l} BC \to \phi(0) = c_3 = 0 \\ \phi(L) = \cancel{c_3} \cosh(L) + c_4 \underbrace{\sinh(L)}_{\neq 0} = 0 \leftrightarrow c_4 = 0 \end{array}\right\} \to c_3 = c_4 = 0.$$

Thus, $\lambda = -\alpha^2 < 0$ is not an eigenvalue.

In conclusion, the summary of step "4. BVP" is:

$$\left.\begin{array}{l}\phi''(x) + \lambda\phi = 0\\ \phi(0) = \phi(L) = 0\end{array}\right\}$$ only has positive eigenvalues, given by

$$\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2}, \text{ with eigenfunction } \boxed{\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)}.$$

5. Product solutions.

We have obtained that

$u(x,t) = \phi(x)G(t)$ is a solution of $\begin{cases} u_t = k\,u_{xx} \\ u(0,t) = u(L,t) = 0 \end{cases}$, if

$$\left.\begin{array}{l} G(t) = c\,e^{-\lambda k t} \\ \phi(x) = \sin(\sqrt{\lambda}\,x) \end{array}\right| \text{ with } \lambda = \left(\frac{n\pi}{L}\right)^2,\ n = 1, 2, \ldots$$

That is, $$\boxed{u(x,t) = \underbrace{B}_{\text{arbitrary constant}} \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}, \quad n = 1, 2, \ldots$$

Remark: We haven't take into account the initial conditions

$$u(x,0) = f(x).$$

6. Initial Value Problem (IVP).

Right now, our solution does not satisfy the initial condition:

$u(x,0) = f(x) \neq B \sin\left(\frac{n\pi x}{L}\right)$ (unless $f(x)$ happens to be a multiple of $\sin\left(\frac{n\pi x}{L}\right)$).

Example: Solve

$$\left.\begin{array}{l} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 4\sin\left(\frac{3\pi x}{L}\right) + 7\sin\left(\frac{8\pi x}{L}\right) \end{array}\right\}$$

Sol:

Notice that by linearity ("superposition principle"), we can split the problem in two:

$$\left.\begin{array}{l} u_{1t} = k u_{1xx} \\ u_1(0,t) = u_1(L,t) = 0 \\ u_1(x,0) = 4\sin\left(\frac{3\pi x}{L}\right) \end{array}\right\} \qquad \left.\begin{array}{l} u_{2t} = k u_{2xx} \\ u_2(0,t) = u_2(L,t) = 0 \\ u_2(x,0) = 7\sin\left(\frac{8\pi x}{L}\right) \end{array}\right\}$$

The solution to the original problem is $u(x,t) = u_1(x,t) + u_2(x,t)$.

Now, we have the solution for each problem:

$$\left.\begin{array}{l} u_1(x,t) = 4\sin\left(\frac{3\pi x}{L}\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t} \\ u_2(x,t) = 7\sin\left(\frac{8\pi x}{L}\right) e^{-k\left(\frac{8\pi}{L}\right)^2 t} \end{array}\right\} \Rightarrow u = u_1 + u_2 \,/\!\!/.$$

→ Principle of Superposition:

Since the problem was linear and homogeneous, and

$u(x,t) = B \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ is a solution for $n = 1, 2, \dots,$

any linear combination is also a solution:

$$\boxed{u(x,t) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

Remark: This allows for more general initial data:

$$u(x,0) = f(x) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right).$$

- Claim: [Chapter 3] (Fourier series)

"Any" function can be written an infinite linear combination of sines,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Therefore, the solution to our problem is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$$

7. Finding B_n: Orthogonality of Sines.

Under the assumption of the previous claim, how do we find the constants B_n so that
$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) ?$$

That is, given $f(x)$, how to compute B_n?

- **Proposition**: Orthogonality of the (eigenfunctions) $\sin\left(\frac{n\pi x}{L}\right)$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n. \end{cases} \qquad n, m = 1, 2, 3, \ldots$$

Proof: Homework.

Let's see how to use this property to find B_n.

Intuition: The functions $\sin\left(\frac{n\pi x}{L}\right)$ are the vectors of an orthogonal basis, and $f(x)$ is written as a linear combination of the basis vectors. The dot product here is defined as the integral over $[0, L]$.

$$\left.\begin{array}{l}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \\ \vec{v}_i \cdot \vec{v}_j = 0 \quad i \neq j\end{array}\right\} \quad \vec{\omega} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n \Rightarrow c_j = \frac{\vec{\omega}\cdot\vec{v}_j}{\vec{v}_j\cdot\vec{v}_j}.$$

$$\uparrow$$
$$\vec{\omega}\cdot\vec{v}_j = c_j\,\vec{v}_j\cdot\vec{v}_j$$

We start with

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Multiply by $\sin\left(\frac{m\pi x}{L}\right)$: $f(x)\sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$.

Integrate from $x=0$ to $x=L$:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{=\begin{cases} 0 & \text{if } m\neq n \\ \frac{L}{2} & \text{if } m=n. \end{cases}} \Rightarrow$$

$$\Rightarrow \int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = B_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx \Rightarrow$$

$$\Rightarrow B_m = \frac{\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = B_m \quad \|.$$