MATH 241 S2020 HW 2 Solutions
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Solutions

Color Key

Important information

Important intermediate step

Final answer

1.4.1

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad u(0)=0 \qquad u(L)=T$$

a)

$$U_E(x) = c_1 x + c_2$$

$$U_E(0) = 0 = c_2$$

$$U_E(L) = c_1 L = T$$

$$c_1 = \frac{T}{L}$$

$$\boxed{U_E(x) = \frac{T}{L}x}$$

d) $u(0)=T \qquad \frac{\partial u}{\partial x}(L) = \alpha$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \Rightarrow 0 = \frac{d^2 u_E}{dx^2}$$

$$U_E(x) = c_1 x + c_2$$

$$U_E(0) = T = c_2$$

$$\frac{du_E}{dx}(L) = \alpha = c_1 L$$

$$c_1 = \frac{\alpha}{L}$$

$$\boxed{U_E(x) = \frac{\alpha}{L}x + T}$$

f) $\frac{Q}{K_0} = x^2 \qquad u(0)=T \qquad \frac{\partial u}{\partial x}(L) = 0$

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + K_0 x^2$$

$$0 = \frac{d^2 u_E}{dx^2} + x^2 \Rightarrow \int\frac{d^2 u_E}{dx^2} = -\int x^2$$

$$\frac{du_E}{dx} = -\frac{1}{3}x^3 + c_1$$

$$\frac{du_E}{dx}(L) = -\frac{1}{3}L^3 + c_1 = 0$$

$$c_1 = \frac{1}{3}L^3$$

$$u_E(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3 x + c_2$$

$$u_E(0) = T = c_2$$

$$\boxed{u_E(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3 x + T}$$

h)

* **(h)** $Q = 0,\qquad \frac{\partial u}{\partial x}(0) - [u(0) - T] = 0,\qquad \frac{\partial u}{\partial x}(L) = \alpha$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$$\Downarrow$$

$$u_E(x) = c_1 x + c_2$$

$$\frac{du_E}{dx}(x) = c_1$$

$$\frac{du_E}{dx}(L) = \alpha = c_1$$

$$\underbrace{\cancel{\frac{du_E}{dx}}}^{\alpha}(0) - [\underbrace{\cancel{u}}^{c_2}(0) - T] = 0$$

$$c_2 = T + \alpha$$

$$\boxed{u_E(x) = \alpha x + T + \alpha}$$

1.4.4

$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L K_0 \frac{\partial^2 u}{\partial x^2}\,dx$$

$$\int_0^L \frac{\partial u}{\partial t}\,dx = K_0\left[\cancel{\frac{\partial u}{\partial x}(L)}^{\,0} - \cancel{\frac{\partial u}{\partial x}(0)}^{\,0}\right]$$

$$\frac{d}{dt}\int_0^L u\,dx = 0$$

$$\boxed{\int_0^L u\,dx = c_1}$$

QED

1.4.7. For the following problems, determine an equilibrium temperature distribution (if one exists). For what values of β are there solutions? Explain physically.

$*$ (a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1, \qquad u(x,0) = f(x), \qquad \dfrac{\partial u}{\partial x}(0,t) = 1, \qquad \dfrac{\partial u}{\partial x}(L,t) = \beta$

total heat energy conserved!

1.4.7

a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 1$

Steady state:

$$0 = \int \frac{d^2 u_E}{dx^2} + 1\,dx$$

$$0 = \frac{du_E}{dx} + x + c_1$$

$$\int \frac{du_E}{dx} = \int -x + c_1$$

c_1 is arbitrary constant

$$u_E(x) = -\frac{1}{2}x^2 + c_1 x + c_2$$

Apply BCs:

$$\frac{du_E}{dx} = -x + c_1$$

$$\frac{du_E}{dx}(0) = 1 = c_1$$

$$\frac{du_E}{dx}(L) = \beta = -L + 1$$

Recall:

Total heat energy $= c\rho A \int u(x,t)\,dx$

Initial heat:

$$c\rho A \int_0^L u(x,0)\,dx = c\rho A \int_0^L f(x)\,dx$$

Final heat:

$$u_E(x) = -\frac{1}{2}x^2 + x + c_2$$

$$c\rho A \int_0^L u_E(x)\,dx = c\rho A\left[\Big|_0^L -\frac{1}{6}x^3 + \frac{1}{2}x^2 + c_2 x\right]$$

$$= c\rho A\left[-\frac{1}{6}L^3 + \frac{1}{2}L^2 + c_2 L\right]$$

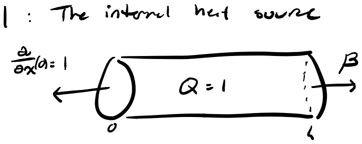
$\therefore$ initial heat = final (solve for c_2)

$$\int_0^L f(x)\,dx = -\frac{1}{6}L^3 + \frac{1}{2}L^2 + c_2 L$$

$$\boxed{c_2 = \frac{1}{6}L^2 - \frac{1}{2}L + \frac{1}{L}\int_0^L f(x)\,dx} \Leftarrow \boxed{u_E(x) = -\frac{1}{2}x^2 + x + c_2}$$

Physical meaning of β?

$Q = 1$: The internal heat source



β is the rate of heat flow at L that counteracts the loss of heat at 0 & the internal generation of heat to maintain the total heat energy in the rod.

c)

Total heat energy is conserved

(c) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x - \beta, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0$

Steady state:

$$0 = \int \frac{d^2 u_E}{dx^2} + x - \beta$$

$$0 = \frac{du_E}{dx} + \frac{1}{2}x^2 - \beta x + c_1$$

$$\frac{du_E}{dx} = -\frac{1}{2}x^2 + \beta x + c_1$$

$$\frac{du_E}{dx}(0) = 0 = c_1$$

$$\frac{du_E}{dx}(L) = 0 = -\frac{1}{2}L^2 + \beta L$$

$$\boxed{\frac{1}{2}L = \beta}$$

$$u_E(x) = -\frac{1}{6}x^3 + \beta\frac{1}{2}x^2 + c_2$$

Final heat:

$$c\rho A \int_0^L -\frac{1}{6}x^3 + \beta\frac{1}{2}x^2 + c_2 \, dx$$

Initial Heat:

$$c\rho A \int_0^L f(x)\, dx$$

$$c\rho A\left[-\frac{1}{24}L^4 + \cancel{\beta}\cancel{\frac{1}{6}}L^3 + c_2 L\right] \qquad \frac{2}{24}\,\frac{1}{12}L^4$$

equate

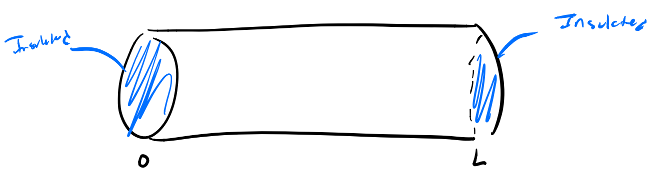
$$\boxed{c_2 = -\frac{1}{24}L^3 + \frac{1}{L}\int_0^L f(x)\,dx}$$

$$\boxed{u_E(x) = -\frac{1}{6}x^3 + \frac{1}{4}x^2 L - \frac{1}{24}L^3 + \frac{1}{L}\int_0^L f(x)\,dx}$$

What is β?

$$Q = x - \beta$$

★ we need to conserve total heat energy to have an equilibrium temp distribution



$\therefore \beta$ be a heat source/sink such that the total internal heat energy generated inside the rod $= 0$ (since boundaries are insulated)

↳ this will allow total heat energy to stay constant!

1.4.10

1.4.10. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = 5$, $\frac{\partial u}{\partial x}(L,t) = 6$. Calculate the total thermal energy in the one-dimensional rod (as a function of time).

Total heat energy: $\int \frac{\partial}{\partial t} dx$

$u(x,0) = f(x)$

$\frac{\partial u}{\partial x}(0,t) = 5$, $\frac{\partial u}{\partial x}(L,t) = 6$

$$\int_0^L \frac{\partial u}{\partial t} dx = \int_0^L \frac{\partial^2 u}{\partial x^2} + 4 \, dx$$

$$= \Big|_0^L \frac{\partial u}{\partial x} + 4x$$

$$= \left(\frac{\partial u}{\partial x}(L) + 4L\right) - \left(\frac{\partial u}{\partial x}(0)\right)$$

$$= (6 + 4L) - 5 = \underline{4L + 1} = \int_0^L \frac{\partial}{\partial t} dx$$

$$c_1 + (4L+1)t = \int_0^L u \, dx$$

$$c_1 = \int_0^L f(x) \, dx$$

* Recall: Total Thermal energy $= \int e(x,t) A(x) \, dx = \int u(x,t) c\rho A(x) dx = E(t)$

$$\boxed{E(t) = c\rho A\left[(4L+1)t + \int_0^L f(x) dx\right]}$$

1.4.11

1.4.11. Suppose $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial x}(0,t) = \beta$, $\frac{\partial u}{\partial x}(L,t) = 7$.

(a) Calculate the total thermal energy in the one-dimensional rod (as a function of time).

(b) From part (a), determine a value of β for which an equilibrium exists. For this value of β, determine $\lim_{t\to\infty} u(x,t)$.

* Recall: Total heat energy $= c\rho A \int u(x,t)\,dx$

a)
$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2} + x \;dx$$

$$= \left.\frac{\partial u}{\partial x} + \frac{1}{2}x^2\right|_0^L$$

$$= \overset{7}{\cancel{\frac{\partial u}{\partial x}}}(L) + \frac{1}{2}L^2 - \overset{\beta}{\cancel{\frac{\partial u}{\partial x}}}(0)$$

$$\frac{d}{dt}\int_0^L u(x,t)\,dx = 7 + \frac{1}{2}L^2 - \beta$$

$$\int_0^L u(x,t)\,dx = \left(7 + \frac{1}{2}L^2 - \beta\right)t + c_2$$

$$\boxed{E(t) = c\rho A\left[\left(7 + \frac{1}{2}L^2 - \beta\right)t + \int_0^L f(x)\,dx\right]}$$

Initial heat = final heat

$$\int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx = c_2$$

b)
$$\frac{d}{dt}\int_0^L u(x,t)\,dx = 7 + \frac{1}{2}L^2 - \beta$$

In order for equilibrium to exist, the LHS must $=0$. (Energy doesn't Δ w/ time)

$$\therefore \beta = \left(7 + \frac{1}{2}L^2\right)$$

$$\Downarrow$$

$$E(t) = c\rho A \int_0^L f(x)\,dx \quad \text{(constant!)}$$

Initial energy = final energy

$$E(0) = c\cancel{\rho}A \int_0^L f(x)\,dx = \cancel{c\rho A}\int_0^L u\,dx$$

$$\downarrow u_E$$

$$= \int_0^L -\frac{1}{6}x^3 + c_1 x + c_2\,dx$$

Steady state:

$$0 = \frac{d^2 u_E}{dx^2} + x$$

$$\int \frac{d^2 u_E}{dx^2} = \int -x$$

$$\int \frac{du_E}{dx} = \int -\frac{1}{2}x^2 + c_1$$

$$u_E(x) = -\frac{1}{6}x^3 + c_1 x + c_2$$

$$\frac{du_E}{dx}(0) = \beta = c_1$$

$$\int_0^L f(x)\,dx = \int_0^L -\frac{1}{6}x^3 + \beta x + c_2$$

$$= -\frac{1}{24}L^4 + \frac{\beta}{2}L^2 + c_2 L$$

$$c_2 = \frac{1}{L}\int_0^L f(x)\,dx + \frac{1}{24}L^3 - \frac{(7+\frac{1}{2}L^2)}{2}L$$

$$\boxed{u_E(x) = -\frac{1}{6}x^3 + \left(7 + \frac{1}{2}L^2\right)x + \frac{1}{L}\int_0^L f(x)\,dx + \frac{1}{24}L^3 - \frac{(7+\frac{1}{2}L^2)}{2}L}$$