Color Key

Important information

Important intermediate step

Final answer

Stuff to pay attention to

HOMEWORK ASSIGNMENT 3

Name: **Due:** Thursday February 6, 10am

1. Determine how β must be related to L if a time-independent (equilibrium) temperature distribution is to exist for the following PDE

$$u_t(x,t) = u_{xx}(x,t) + x - \beta,$$

for $0 < x < L$ and $t > 0$, with boundary conditions

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0,$$

for $t \geq 0$. Is this equilibrium solution unique for the value of β determined? Explain.

2. Consider the heat equation $u_t = k u_{xx}$ for $0 < x < L$, subject to the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$ for $t \geq 0$. Solve the initial value problem if the temperature is initially

a) $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$,

b) $u(x,0) = \begin{cases} 1, & 0 < x \leq L/2, \\ 2, & L/2 < x < L. \end{cases}$

3. Evaluate

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx, \qquad \text{for } n \geq 0, m \geq 0.$$

Hint: Use the trigonometric identity

$$\cos(a)\cos(b) = \frac{1}{2}\big(\cos(a+b) + \cos(a-b)\big).$$

4. Consider $u_t = k u_{xx} - \alpha u$. Suppose that the boundary conditions are $u(0,t) = u(L,t) = 0$ for $t \geq 0$.

a) What are the possible equilibrium solutions if $\alpha > 0$?

b) Solve the time-dependent problem with $u(x,0) = f(x)$ if $\alpha > 0$. Find the limit as $t \to \infty$ and compare to part a).

5. Consider the PDE $u_x(x,y) = x u_y(x,y)$, for $a < x < b$ and $c < y < d$. If non-zero solutions of the form $u(x,y) = \alpha(x)\beta(y)$ are to be constructed for this PDE, then determine up to three arbitrary constants the forms of $\alpha(x)$ and $\beta(y)$.

6. Decide whether the following statements are true or false:

   a) Consider the heat equation $u_t = u_{xx}$ on a one-dimensional rod of length L subject to the boundary conditions

   $$u(0,t) = 5, \qquad u(L,t) = 8.$$

   The sum of any two solutions (to the given heat equation and boundary conditions) is again a solution.

   b) The method of separation of variables (when it works) solves a PDE by converting it into ordinary differential equations.

7. Solve the heat equation $u_t = 2u_{xx}$, $0 < x < 1$, $t > 0$, subject to $u_x(0,t) = u_x(1,t) = 0$ for $t \geq 0$, in the following cases:

   a) $u(x,0) = 2 + 7\cos(4\pi x)$,

   b) $u(x,0) = -2\sin(\pi x)$.

8. Consider the heat equation $u_t = 3u_{xx}$ where $0 < x < 1$ and $t > 0$, with boundary conditions $u(0,t) = 0 = u_x(1,t)$, and initial condition

   $$u(x,0) = 4\sin\left(\frac{3\pi x}{2}\right) + 7\sin\left(\frac{5\pi x}{2}\right).$$

   a) What is the physical meaning of the boundary condition at the left end-point $x = 0$?

   b) Is there any heat source?

   c) Solve the above initial boundary value problem. You may assume for this problem that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

9. Read Sections (2.5), (3.1), (3.2) and (3.3).

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1

1. Determine how β must be related to L if a time-independent (equilibrium) temperature distribution is to exist for the following PDE

$$u_t(x,t) = u_{xx}(x,t) + x - \beta,$$

for $0 < x < L$ and $t > 0$, with boundary conditions

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0,$$

for $t \geq 0$. Is this equilibrium solution unique for the value of β determined? Explain.

$$\int_0^L \frac{\partial u}{\partial t}\,dx = \int_0^L \frac{\partial^2 u}{\partial x^2} + x - \beta \,dx$$

$$\frac{d}{dt}\int_0^L u\,dx = \Big|_0^L \frac{\partial u}{\partial x} + \frac{1}{2}x^2 - \beta x$$

$$= \left(\cancel{\frac{\partial u}{\partial x}(L)}^{\,0} + \frac{1}{2}L^2 - \beta L\right) - \left(\cancel{\frac{\partial u}{\partial x}(0)}^{\,0}\right)$$

$$= \frac{1}{2}L^2 - \beta L = \underline{0}$$ } total heat energy cannot change w/ time

$$\beta = \frac{1}{2}L$$

Steady state:

$$0 = \frac{d^2 u_E}{dx^2} + x - \beta$$

$$\int \frac{d^2 u_E}{dx^2} = \int -x + \beta$$

$$\frac{du_E}{dx} = -\frac{1}{2}x^2 + \beta x + c_1 \Rightarrow c_1 = 0 \quad (\text{via BCs})$$

$$\boxed{u_E(x) = -\frac{1}{6}x^3 + \frac{\beta}{2}x^2 + c_2}$$ → can't determine c_2 since no IC given

This $u_E(x)$ is **not** unique for the value of β because an initial condition to find the value of c_2 isn't given.

If an IC were given, then the solution would be unique

2. Consider the heat equation $u_t = ku_{xx}$ for $0 < x < L$, subject to the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$ for $t \geq 0$. Solve the initial value problem if the temperature is initially

a) $u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right)$,

b) $u(x,0) = \begin{cases} 1, & 0 < x \leq L/2, \\ 2, & L/2 < x < L. \end{cases}$

PDE: $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$

BCs: $\left.\begin{matrix} u(0,t)=0 \\ u(L,t)=0 \end{matrix}\right\}$ Dirichlet BCs

Solving general solution

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

1) Sep of variables

let $u(x,t) = \phi(x)h(t)$

$$\frac{1}{k}\frac{1}{h}\frac{dh}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

2) Solve ODEs

$h(t)$:

$$\frac{dh}{dt} = -\lambda k h \Rightarrow h(t) = c_1 e^{-\lambda k t}$$

$\phi(x)$:

Test all 3 possible cases for λ!

$\lambda > 0$:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \Rightarrow \phi(x) = c_1\sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x)$$

$$\phi(0) = 0 = c_2$$

$$\phi(L) = 0 = c_1\sin(\sqrt{\lambda}L)$$

$$\sqrt{\lambda}L = n\pi$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \Rightarrow \phi(x) = c_1\sin\left(\frac{n\pi x}{L}\right)$$

$\lambda = 0$:

$$\frac{d^2\phi}{dx^2} = 0 \Rightarrow \phi(x) = c_1 x + c_2$$

$$\phi(0) = 0 = c_2$$

$$\phi(L) = 0 = c_1 L$$

c_1 must $= 0$!

$$\therefore \phi(x) \equiv 0 \Rightarrow \text{trivial}$$

$$\therefore \lambda \neq 0$$

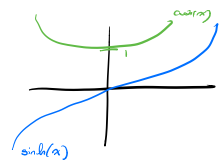
$\lambda < 0$:

λ is inherently a $(-)$ number

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

redefine

let $-\lambda = s$



$$\frac{d^2\phi}{dx^2} = s\phi \Rightarrow \phi(x) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$$

$$\phi(x) = c_1 \sinh(\sqrt{s}x) + c_2 \cosh(\sqrt{s}x)$$

$$\phi(0) = 0 = c_2$$

$$\phi(L) = c_1 \underbrace{\sinh(\sqrt{s}L)}_{\text{positive number}} = 0$$

c_1 must be 0

$\Downarrow$

$$\therefore \phi(x) \equiv 0 \quad (\text{Trivial solution})$$

$\therefore \lambda$ cannot be < 0

After casing, $\lambda > 0$

③ Product sols & superposition

$$u(x,t) = \phi(x)h(t)$$

$$= \sum_{n=1}^{\infty} \underbrace{A_n \sin\left(\frac{n\pi x}{L}\right)}_{\phi(x)} \underbrace{e^{-\lambda_n k t}}_{h(t)}$$

a)

$$u(x,0) = 6\sin\left(\frac{9\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$\Downarrow$ coefficient matching

$n=9$: $A_9 = 6$

$$\boxed{u(x,t) = 6\sin\left(\frac{9\pi x}{L}\right) e^{-\left(\frac{6\pi}{L}\right)^2 kt}}$$

b)

$$u(x,0) = \begin{cases} 1 & 0 < x \le \frac{L}{2} \\ 2 & \frac{L}{2} < x < L \end{cases}$$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

i) multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$

$$u(x,0) = \begin{cases} 1\sin\left(\frac{m\pi x}{L}\right) & 0 < x \le \frac{L}{2} \\ 2\sin\left(\frac{m\pi x}{L}\right) & \frac{L}{2} < x < L \end{cases} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

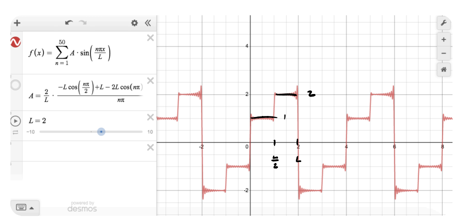
ii) integrate both sides

↳ split integral accordingly

$$\int_0^{\frac{L}{2}} \sin\left(\frac{m\pi x}{L}\right)dx + 2\int_{\frac{L}{2}}^{L} \sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

iii) $m=n$ is the only surviving term; solve for A

$$A_n = \frac{\int_0^{\frac{L}{2}} \sin\left(\frac{m\pi x}{L}\right)dx + 2\int_{\frac{L}{2}}^{L} \sin\left(\frac{m\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx}$$



$$= \left(-\frac{L\left(\cos\left(\frac{n\pi}{2}\right)-1\right)}{\pi n} - \frac{2L\left(\cos(\pi n) - \cos\left(\frac{\pi n}{L}\right)\right)}{\pi n}\right)\frac{2}{L}$$

$$\boxed{A_n = \frac{-L\cos\left(\frac{n\pi}{2}\right) + L - 2L\cos(\pi n) + 2L\cos\left(\frac{\pi n}{L}\right)}{\pi n}}$$

3

3. Evaluate
$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx, \qquad \text{for } n \geq 0, m \geq 0.$$

$$\begin{cases} n \neq m = 0 \\ n = m = \frac{L}{2} \end{cases}$$

Hint: Use the trigonometric identity
$$\cos(a)\cos(b) = \frac{1}{2}\big(\cos(a+b) + \cos(a-b)\big).$$

Apply identity $(n \neq m)$

$$\frac{1}{2}\int_0^L \cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right) dx$$

$$\frac{1}{2}\Big|_0^L (\text{consts})\sin\left(\frac{(n+m)\pi x}{L}\right) + (\text{consts})\sin\left(\frac{(n-m)\pi x}{L}\right)$$

$$(\text{consts})\sin\left(\frac{(n+m)\pi L}{L}\right) + (\text{consts})\sin\left(\frac{(n-m)\pi L}{L}\right)$$

$$(\text{consts})\underbrace{\sin\big((n+m)\pi\big)}_{\text{always } 0} + (\text{consts})\underbrace{\sin\big((n-m)\pi\big)}_{\text{always } 0} = \boxed{0}$$

$n = m$:

$$\int_0^L \cos^2\left(\frac{n\pi x}{L}\right)dx = \frac{L(\cancel{\sin(2\pi n)}^{0} + 2\pi n)}{4\pi n} = \boxed{\frac{L}{2}}$$

$n = m = 0$:

$$\int_0^L dx = \boxed{L}$$

4

4. Consider $u_t = ku_{xx} - \alpha u$. Suppose that the boundary conditions are $u(0,t) = u(L,t) = 0$ for $t \geq 0$.

a) What are the possible equilibrium solutions if $\alpha > 0$?

b) Solve the time-dependent problem with $u(x,0) = f(x)$ if $\alpha > 0$. Find the limit as $t \to \infty$ and compare to part a).

a) Equilibrium approaches 0. Heat will dissipate.

$$0 = k\frac{d^2u_E}{dx^2} - \alpha u_E$$

$$k\frac{d^2u_E}{dx^2} = \alpha u_E \Rightarrow \frac{d^2u_E}{dx^2} = \left(\frac{\alpha}{k}\right)u \quad (\text{This is second order ODE})$$

$$u_E(x) = c_1 \sinh\left(\sqrt{\frac{\alpha}{k}}\, x\right) + c_2 \cosh\left(\sqrt{\frac{\alpha}{k}}\, x\right)$$

$$u_E(0) = 0 = c_2$$

$$u_E(L) = 0 = \underbrace{c_1}_{=0} \underbrace{\sinh\left(\overbrace{\sqrt{\frac{\alpha}{k}}\, L}^{\text{nonzero}}\right)}_{\text{nonzero}}$$

$$\boxed{\therefore\ u_E(x) = 0}$$

b) $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u$$

1) separation of variables

$$u(x,t) = \phi(x)\, h(t)$$

$$\frac{d}{dt}[\phi h] = k\frac{d^2}{dx^2}[\phi h] - \alpha[\phi h]$$

$$\frac{1}{h}\frac{dh}{dt} = k\frac{1}{\phi}\frac{d^2\phi}{dx^2} - \alpha$$

$$\frac{1}{k}\frac{1}{h}\frac{dh}{dt} + \frac{\alpha}{k} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

2) solve ode's

$h(t)$:

$$\frac{dh}{dt} = -(\lambda k + \alpha)\, h$$

$$h(t) = c_1 e^{-(\lambda k + \alpha)t}$$

$\phi(x)$:

Same casing as above in Problem 2

$$\lambda = \left(\frac{n\pi}{L}\right)^2,\ n = 1, 2\ldots \Rightarrow \phi(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

3) prod sols

$$u(x,t) = \sum_{n=1}^{\infty} A_n \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\phi(x)} \underbrace{e^{-(\lambda_n k + \alpha)t}}_{h(t)}$$

4) Solve for A_n

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{A_n = \frac{\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)}}$$

5) put all togethr

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\lambda_n k + \alpha)t}}$$

$$\boxed{\lim_{t\to\infty} u(x,t) = 0 \;!}$$

Limit of $\lim_{t\to\infty} u(x,t) = 0 \Rightarrow$ matches u_E in part (a)

5

5. Consider the PDE $u_x(x,y) = xu_y(x,y)$, for $a < x < b$ and $c < y < d$. If non-zero solutions of the form $u(x,y) = \alpha(x)\beta(y)$ are to be constructed for this PDE, then determine up to three arbitrary constants the forms of $\alpha(x)$ and $\beta(y)$.

$$\frac{1}{x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

1) sep of vars

$$u(x,t) = \phi(x)\, h(y)$$

$$\frac{1}{x}\frac{1}{\phi}\frac{d\phi}{dx} = \frac{1}{h}\frac{dh}{dy} = -\lambda$$

2) solve ODE's

h(y):

$$\frac{dh}{dy} = -\lambda h \Rightarrow h(y) = c_1 e^{-\lambda y}$$

$\phi(x)$:

$$\frac{d\phi}{dx} = -\lambda x \phi \Rightarrow \phi(x) = c_2 e^{-\frac{1}{2}\lambda x^2}$$

3) Prod sols

6

6. Decide whether the following statements are true or false:

a) Consider the heat equation $u_t = u_{xx}$ on a one-dimensional rod of length L subject to the boundary conditions

$$u(0,t) = 5, \qquad u(L,t) = 8.$$

False

The sum of any two solutions (to the given heat equation and boundary conditions) is again a solution.

b) The method of separation of variables (when it works) solves a PDE by converting it into ordinary differential equations. True

a) FALSE!

This is actually a very interesting question.

In order to have a linear combination of solutions, the PDE & BCs must both be homogeneous.

Let:

$v_t = k v_{xx}$ $\qquad$ $w_t = k w_{xx}$

$v(0,t) = 5$ $\qquad$ $w(0,t) = 5$

$v(L,t) = 8$ $\qquad$ $w(L,t) = 8$

$\therefore$ $u = v + w$ should be a sol as well, not

$u(0,t) = v(0,t) + w(0,t) = 10$ ✗

$u(L,t) = v(L,t) + w(L,t) = 16$ ✗

NOT a solution to original PDE

7

. Solve the heat equation $u_t = 2u_{xx}$, $0 < x < 1$, $t > 0$, subject to $\underbrace{u_x(0,t) = u_x(1,t) =}_{\text{Neumann BCs}}$ 0 for $t \geq 0$, in the following cases:

a) $u(x,0) = 2 + 7\cos(4\pi x)$,

b) $u(x,0) = -2\sin(\pi x)$.

1) Sep. of vars

$$u_t = 2u_{xx}$$

$$\text{let } u(x,t) = \phi(x)h(t)$$

$$\frac{1}{2}\frac{1}{h}h_t = \frac{1}{\phi}\phi_{xx} = -\lambda$$

2) Solve ODEs

$h(t)$:

$$h_t = -2\lambda h \Rightarrow h(t) = c_1 e^{-2\lambda t}$$

$\phi(x)$:

$\lambda > 0$:

$$\phi_{xx} = -\lambda \phi \Rightarrow \phi(x) = c_2 \sin(\sqrt{\lambda}\, x) + c_3 \cos(\sqrt{\lambda} x)$$

$$\phi_x = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x) - c_3 \sqrt{\lambda} \sin(\sqrt{\lambda} x)$$

$$\phi_x(0) = 0 = c_2$$

$$\phi_x(1) = 0 = -c_3 \sqrt{\lambda} \sin(\sqrt{\lambda})$$

$$\therefore \lambda = (n\pi)^2$$

$$\phi(x) = c_3 \cos(n\pi x)$$

$\lambda = 0$:

$$\phi_{xx} = 0 \Rightarrow \phi(x) = c_4 x + c_5$$

$$\phi_x = c_4$$

$$\phi_x(0) = 0 = c_4$$

$$\phi_x(1) = 0 = c_4$$

$$\therefore \phi(x) = c_5 \Rightarrow \lambda = 0 \checkmark$$

$\underline{\lambda < 0:}$

λ is a number less than 0!

$$\phi_{xx} = -\lambda\phi$$

let $s = -\lambda$

$$\phi_{xx} = s\phi \Rightarrow \phi(x) = c_6 \sinh(\sqrt{s}\,x) + c_7 \cosh(\sqrt{s}\,x)$$

$$\phi_x = c_7\sqrt{s}\cosh(\sqrt{s}\,x) + c_7\sqrt{s}\sinh(\sqrt{s}\,x)$$

$$\phi_x(0) = \underline{0 = c_7}$$

$$\phi_x(1) = 0 = \underbrace{c_7}_{\text{must be } 0}\underbrace{\sqrt{s}}_{\text{nonzero}}\underbrace{\sinh(\sqrt{s})}_{\text{nonzero}}$$

$$\therefore \phi(x) \equiv 0 \Rightarrow \text{TRIVIAL}$$

$\therefore$ $\underline{\lambda \text{ cannot be } < 0!}$

3) Prod sols & superpos

$$u(x,t) = \phi(x)\,h(t)$$

$$= \sum_{n=0}^{\infty} A_n \cos(n\pi x)\, e^{-2\lambda_n t}$$

note the index!

4) Find A_n

Q7a) $u(x,0) = 2 + 7\cos(4\pi x)$

* expand out first term in series

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = 2 + 7\cos(4\pi x)$$

$\Downarrow$ coeff. matching (via orthogonality)

$n=0: \quad A_0 = 2$

$n=4: \quad A_4 = 7$

$$\boxed{u(x,t) = 2 + 7\cos(4\pi x)\, e^{-2(7\pi)^2 t}}$$

Q7b) $u(x,0) = -2\sin(\pi x)$

* apply general orthogonality rules

$$u(x,0) = -2\sin(\pi x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

$$\int_0^1 -2\sin(\pi x)\cos(m\pi x)\,dx = \underbrace{\int_0^1 A_0 \cos(m\pi x)\,dx}_{m=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos(n\pi x)\cos(m\pi x)\,dx}_{m=n}$$

$m=0: \quad \int_0^1 -2\sin(\pi x)\,dx = A_0 \int_0^1 dx$

$$-\frac{4}{\pi} = A_0$$

$m=n$: $$\int_0^1 -2\sin(\pi x)\cos(n\pi x)\,dx = A_n\int_0^1 \cos^2(n\pi x)\,dx$$

$$\frac{2((-1)^n+1)}{\pi(n^2-1)} = A_n\left(\frac{1}{2}\right)$$

$$A_n = \frac{4((-1)^n+1)}{\pi(n^2-1)}$$

$$u(x,t) = -\frac{4}{\pi} + \sum_{n=1}^{\infty}\left[\frac{4((-1)^n+1)}{\pi(n^2-1)}\right]\cos(n\pi x)$$

Q8

8. Consider the heat equation $u_t = 3u_{xx}$ where $0 < x < 1$ and $t > 0$, with boundary conditions $\underbrace{u(0,t) = 0 = u_x(1,t)}_{\text{Mixed BCs!}}$, and initial condition

$$u(x,0) = 4\sin\left(\frac{3\pi x}{2}\right) + 7\sin\left(\frac{5\pi x}{2}\right).$$

a) What is the physical meaning of the boundary condition at the left end-point $x = 0$?

b) Is there any heat source?

c) Solve the above initial boundary value problem. You may assume for this problem that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

a) At $x=0$, we are in a water bath at temperature 0

b) No heat source; $Q=0$

c) Similar setup to (Q7)

1) Sep. of vars

$$u_t = 3u_{xx}$$

let $u(x,t) = \phi(x)h(t)$

$$\frac{1}{3}\frac{1}{h}h_t = \frac{1}{\phi}\phi_{xx} = -\lambda$$

2) Solve ODEs

$h(t)$:

$$h_t = -3\lambda h \Rightarrow h(t) = c_1 e^{-3\lambda t}$$

$\phi(x)$:

$\lambda > 0$:

$$\phi_{xx} = -\lambda\phi \Rightarrow \phi(x) = c_2 \sin(\sqrt{\lambda}\,x) + c_3\cos(\sqrt{\lambda}\,x)$$

$$\phi(0) = 0 = c_3$$

$$\phi_x = c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$$

$$\phi_x(1) = 0 = c_2\sqrt{\lambda}\cos(\sqrt{\lambda})$$

zeros of cosine!

$$\lambda = \left(\frac{(2n-1)\pi}{2}\right)^2 \qquad \phi(x) = c_2\sin(\sqrt{\lambda}\,x)$$

$\lambda = 0$:

$$\phi_{xx} = 0 \Rightarrow \phi(x) = c_4 x + c_5$$

$$\left.\begin{aligned} \phi(0) &= 0 = c_5 \\ \phi_x &= c_4 \\ \phi_x(1) &= 0 = c_4 \end{aligned}\right\} \therefore \lambda \neq 0$$

$\lambda < 0$:

λ is a number less than 0!

$$\phi_{xx} = -\lambda\phi$$

let $s = -\lambda$

$$\phi_{xx} = s\phi \Rightarrow \phi(x) = c_6 \sinh(\sqrt{s}\,x) + c_7 \cosh(\sqrt{s}\,x)$$

$$\phi(0) = 0 = c_7$$

$$\phi_x = c_6 \sqrt{s} \cosh(\sqrt{s}\,x)$$

$$\phi_x(1) = c_6 \sqrt{s} \cosh(\sqrt{s}) = 0$$

c_6 must be 0

$$\therefore \phi(x) \equiv 0 \Rightarrow \lambda \text{ is not } < 0!$$

3) Product sols & superpos

$$u(x,t) = \phi(x)\,h(t)$$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi x}{2}\right) e^{-3\lambda_n t}$$

4) Apply IC & Find A_n

$$u(x,0) = 4\sin\left(\frac{3\pi x}{2}\right) + 7\sin\left(\frac{5\pi x}{2}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{(2n-1)\pi x}{2}\right)$$

$\Downarrow$ coefficient match (via orthogonality)

$n=2$: $A_2 = 4$

$n=3$: $A_3 = 7$

5) Put all together

$$\boxed{u(x,t) = 4\sin\left(\frac{3\pi x}{2}\right)e^{-3\left(\frac{3\pi}{2}\right)^2 t} + 7\sin\left(\frac{5\pi x}{2}\right)e^{-3\left(\frac{5\pi}{2}\right)^2 t}}$$