MATH 425 LECTURE 3 : The coordinate method. Fundamental PDEs I.

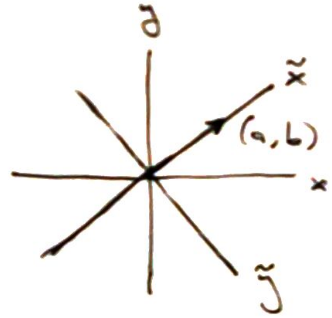
- In the previous lecture we learn that first-order linear PDEs can be solved by the method of characteristics. We introduce now a related method that also solves these PDEs: <u>The coordinate method</u>.

Basically, it consists in applying an appropriate change of variables. We will repeat some examples using this method.

- <u>Example</u>: (Constant coefficient case)

$$a u_x + b u_y = 0$$

<u>Sol</u>: $(a,b)\cdot \nabla u = 0$



Change of variables:

$$\left.\begin{aligned} \tilde{x} &= ax + by \\ \tilde{y} &= bx - ay \end{aligned}\right\}$$ and now apply the chain rule,

$$u_x = \frac{\partial u}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial u}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x} = a\, u_{\tilde{x}} + b\, u_{\tilde{y}},$$

$$u_y = \frac{\partial u}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial y} + \frac{\partial u}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y} = b\, u_{\tilde{x}} - a\, u_{\tilde{y}},$$

so that

$$a\, u_x + b u_y = a^2 u_{\tilde{x}} + \cancel{ab}\, u_{\tilde{y}} + b^2 u_{\tilde{x}} - \cancel{ab}\, u_{\tilde{y}} = (a^2+b^2)\, u_{\tilde{x}} = 0 \Rightarrow$$

$$\Rightarrow \| \; u_{\tilde{x}} = 0.$$

In conclusion, $u = f(\tilde{y}) = f(bx-ay)$ //.

→ Remark: We could have chose $\tilde{x}$ in a different way. The important part was having one of the new variables equal to $bx-ay$ (we know that the solution only depends on this quantity, so the partial derivative with respect to the other variable will be zero).

↳ Exercise (Home) Solve $a u_x + b u_y = 0$ by changing $\begin{cases} \tilde{x} = x \\ \tilde{y} = bx - ay. \end{cases}$

• Example: (Variable coefficient case)

$$x u_x - y u_y + y^2 u = y^2$$ (compare the solution with the example on page -8-, -9-).

Sol:

New coordinates? (i.e., change of variables?).

The characteristic curves are given by

$$\frac{dy}{dx} = \frac{-y}{x} \Rightarrow \log(y) = -\log(x) + c \Rightarrow y = \frac{c}{x} \Rightarrow C = xy.$$

Let's try then with $\begin{cases} \tilde{x} = x & \text{(or something different)}. \\ \tilde{y} = xy \leftarrow \end{cases}$

The chain rules gives that

((*) → See end of page -13-)

$$\left.\begin{aligned} u_x &= u_{\tilde{x}} + y u_{\tilde{y}} \\ u_y &= x u_{\tilde{y}} \end{aligned}\right\} \Rightarrow x u_x - y u_y = x u_{\tilde{x}} + \cancel{xy} u_{\tilde{y}} - \cancel{xy} u_{\tilde{y}} = \underbrace{x u_{\tilde{x}}}_{u_{\tilde{y}} \text{ does not appear}}$$

To write the PDE in $\tilde{x}$ and $\tilde{y}$ we need to solve for x and y:

$$\left.\begin{aligned} x &= \tilde{x} \\ y &= \frac{\tilde{y}}{x} = \frac{\tilde{y}}{\tilde{x}} \end{aligned}\right\}$$

Then, we obtain that

$$\tilde{x} u_{\tilde{x}} + \left(\frac{\tilde{y}}{\tilde{x}}\right)^2 u = \left(\frac{\tilde{y}}{\tilde{x}}\right)^2 \rightarrow u_{\tilde{x}} + \frac{\tilde{y}^2}{\tilde{x}^3} u = \frac{\tilde{y}^2}{\tilde{x}^3}.$$

This can be seen now as a first-order linear ODE for u in $\tilde{x}$:

Integrating factor $e^{\int \frac{\tilde{y}^2}{\tilde{x}^3} d\tilde{x}} = e^{-\frac{\tilde{y}^2}{2\tilde{x}^2}}$,

$$\frac{\partial}{\partial x}\left(e^{-\frac{\tilde{y}^2}{2\tilde{x}^2}} u(\tilde{x},\tilde{y})\right) = \frac{\tilde{y}^2}{\tilde{x}^3} e^{-\frac{\tilde{y}^2}{2\tilde{x}^2}} \Rightarrow$$

$$\Rightarrow u(\tilde{x},\tilde{y}) = e^{\frac{\tilde{y}^2}{2\tilde{x}^2}}\left(\int \frac{\tilde{y}^2}{\tilde{x}^3} e^{\frac{-\tilde{y}^2}{2\tilde{x}^2}} d\tilde{x} + c(\tilde{y})\right) =$$

$$= 1 + c(\tilde{y}) e^{\frac{\tilde{y}^2}{2\tilde{x}^2}}.$$

Going back to x, y,

$$u(x,y) = 1 + c(xy) e^{\frac{x^2y^2}{2x^2}} = 1 + c(xy) e^{y^2/2}.$$

• Remark. The solutions are usually easy to check by direct substitution into the PDE.

• Remark: (*) We have to be careful with the abuse of notation. On page -12- we wrote:

$$\left.\begin{aligned}\tilde{x} &= x\\ \tilde{y} &= xy\end{aligned}\right\} \text{ and thus } \left.\begin{aligned}u_x &= u_{\tilde{x}} + y u_{\tilde{y}}\\ u_y &= x u_{\tilde{y}}\end{aligned}\right\} \text{ so we obtained}$$

$$x u_x - y u_y = x u_{\tilde{x}}.$$

From this notation, one might think "since $x = \tilde{x}$", then

$u_{\tilde{x}} \equiv u_x$ and so $\quad x\cancel{u_x} - y u_y = x\cancel{u_x} \Rightarrow u_y = 0,$

which contradicts our previous solution.

The mistake comes from our notation. Being precise, we should write:

Change of variables: $\left.\begin{aligned} \tilde{x}(x,y) &= x \\ \tilde{y}(x,y) &= xy \end{aligned}\right\}$

and define a new function through the relation

$$u(x,y) = v(\tilde{x}(x,y), \tilde{y}(x,y)).$$

Then, the chain rule says that

$$u_x(x,y) = v_{\tilde{x}}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \tilde{x}}{\partial x}(x,y) + v_{\tilde{y}}(\tilde{x}(x,y), \tilde{y}(x,y)) \frac{\partial \tilde{y}}{\partial x}(x,y),$$

and similarly with y. We would find that

$$x u_x(x,y) - y u_y(x,y) = x v_{\tilde{x}}(\tilde{x}(x,y), \tilde{y}(x,y))$$

now it is clear they are not the same object.

In summary, we obtain a new PDE in v, but we usually abuse of notation and write it with u.

$\blacksquare$

[Section 3.1] Fundamentals PDEs.

We have only study first-order linear PDEs so far.
An important real example of such PDEs comes from modelling the transport of a suspended pollutant in a fluid flow: the transport equation.
However, the three most fundamental or prototypical <u>linear</u> PDEs are of second order: the heat, wave, and Laplace equation.
In this section we will show how to model certain physical problems using PDEs.

<u>1</u>) Derivation of the heat equation.

Physical problem: Consider a rod whose temperature distribution we know at a given time. What equation describes the evolution of the temperature distribution?

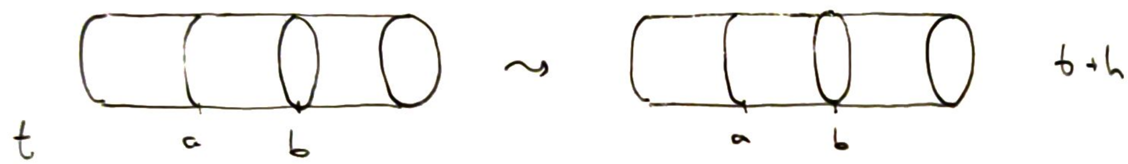
For simplicity, we will make some assumptions:

1) Straight rod with constant section area $A(x) = A$.

2) All properties are constant on each section. That is, there is only dependence in x (1-d problem).

3) Lateral surface is insulated, no heat generation inside.

One fundamental principles of physics (1st law of thermodynamics) tells us that energy is conserved. Consider two arbitrary points a, b, and let's study that region of the rod between a time instant t and another one $t+h$ (with $h>0$),



The conservation of energy then says

" Energy in $[a,b]$ at time $t+h$ (1) = Energy in $[a,b]$ at time t (2) + Energy that has come in between t and $t+h$ (3) − Energy that has come out between t and $t+h$ (4). "

Let's write this mathematically.

Denote $e(x,y,z,t) \equiv$ (thermal) energy density

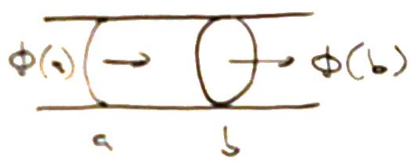
$\Phi(x,y,z,t) \equiv$ (thermal energy) flux $\sim \dfrac{\text{energy}}{\text{area} \times \text{time}}$

$\equiv$ energy flowing <u>to the right</u> per unit area and per unit time

Then,

$$\int_{\substack{\text{Region}\\ [a,b]}} e(x,y,z,t+h)\,dV = \int_{\substack{\text{Region}\\ [a,b]}} e(x,y,z,t)\,dV + \int_t^{t+h}\int_{\substack{\text{section}\\ \text{at } a}} \Phi(a,y,z,\tau)\,dS\,d\tau +$$

$$- \int_t^{t+h}\int_{\substack{\text{section}\\ \text{at } b}} \Phi(b,y,z,\tau)\,dS\,d\tau$$



Using assumptions 1), 2), we obtain that

$$A\int_a^b e(x,t+h)\,dx = A\int_a^b e(x,t)\,dx + A\int_t^{t+h}\Phi(a,\tau)\,d\tau - A\int_t^{t+h}\Phi(b,\tau)\,d\tau .$$

Dividing by h and taking the limit as $h \to 0^+$,

$$\frac{d}{dt}\int_a^b e(x,t)\,dx = \Phi(a,t) - \Phi(b,t) .$$

This can be rewritten as

$$\int_a^b e_t(x,t)\,dx = -\int_a^b \phi_x(x,t)\,dx$$ and because both a, b,

were arbitrary, it must hold that

$$\left| e_t(x,t) = -\phi_x(x,t) \right|$$

- <u>Modelling e and ϕ</u>

Thermodynamics $\Rightarrow$ $e(x,t) = c(x)\,\rho(x)\,u(x,t)$

specific heat — mass density — temperature

<u>Flux</u>: Fourier's law

$$\phi(x) = -\underbrace{K(x)}_{\text{thermal conductivity}}\,u_x(x,t)$$

(This law is less fundamental. It holds under certain conditions).

When $c(x), \rho(x), K(x)$ are all constants, we obtain the so-called heat equation

$\left(k = \frac{K}{c\rho}\right.$ thermal diffusivity$\left.\right)$.

$$\boxed{u_t = k\,u_{xx}}$$ 1-d HEAT EQUATION

<u>Exercise</u> (Home): Check that the diffusion of a dissolved substanted in a fluid at rest satisfies the same PDE (up to renaming the physical constants).

[Instead of Fourier's law, you'll need Fick's law of diffusion:

$\phi = -k\,\rho_x$, where $\phi \equiv$ (mass) flux.]